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ON THE NUMBER OF CONJUGACY CLASSES IN A FINITE GROUP II†

BY

ANTONIO VERA-LÓPEZ[a] AND M[A] CONCEPCIÓN LARREA[b]

Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, Bilbao, Spain; and Departamento de Matemáticas, Facultad de Económicas, Universidad del País Vasco, Apartado 644, Bilbao, Spain

ABSTRACT

In this work we obtain new properties connected with the number of conjugacy classes of elements of a finite group, through the analysis of the number $r_G(gN)$ of conjugacy classes of elements of G that intersect the coset gN, where N is a normal subgroup of G and g any element of G. The results obtained about this number are not only used in the general problem of classifying finite groups according to the number of conjugacy classes, but they also allow us to improve and generalize known results relating to conjugacy classes due to P. Hall, M. Cartwright, A. Mann, G. Sherman, A. Vera-López and L. Ortíz de Elguea. Examples are given which illustrate our improvements.

Introduction

Throughout this paper, G denotes a finite group and π a set of prime numbers. Notation used without further explanation is standard. In addition, $\pi(G)$ denotes the set of all primes dividing the order of $G:|G|$; $r^\pi(G)$ the number of conjugacy classes of π-elements of G and $r(G)=r^{\pi(G)}(G)$. If S is a non-empty subset of G, $[S]_\pi$ denotes the set of all π-elements contained in S, and $r_G^\pi(S)$ the number of conjugacy classes of π-element of G that intersect S. If S is a normal subset of G and $[S]_\pi$ is disjoint union of the conjugacy classes $\mathrm{Cl}_G(x_1),\ldots,\mathrm{Cl}_G(x_s)$, ordered so that $|C_G(x_1)|\geqq\cdots\geqq|C_G(x_s)|$, then we define

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$$_\pi\Delta_S^G = (|C_G(x_1)|, \ldots, |C_G(x_s)|).$$

In particular, $\Delta_G = {}_{\pi(G)}\Delta_G^G$ denotes the conjugacy-vector of G. Let N be a normal subgroup of G. Then for each coset gN, we write ${}_\pi\Delta_{gN}^G$ instead of ${}_\pi\Delta^G_{\cup_{x\in G} g^xN}$ (by abuse of notation!), and we also write $\Delta_{gN}^G = {}_{\pi(G)}\Delta_{gN}^G$, in case $\pi = \pi(G)$. In general, we will use the following convention: when $\pi = \pi(G)$, the symbol π will be dropped from all above notation.

Every element g of G has a unique decomposition $g = g_\pi g_{\pi'} = g_{\pi'} g_\pi$ into a π-element g_π and a π'-element $g_{\pi'}$, where π' denotes the complementary set of primes with respect to π.

If S_1 and S_2 are two non-empty subsets of G, we define

$$T_{S_1,S_2} = \{(x, y) \in S_1 \times S_2 \mid xy = yx\},$$

and if p is a prime number dividing n, then we write $p^a \parallel n$ to indicate that p^a is the greatest power of p dividing n, that is, $p^a \mid n$, but $p^{a+1} \nmid n$. We write $v_p(n) = a$, in this case. Two n-tuples $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n)$ are said to be $\sim$-equivalent, whenever $\{u_1, \ldots, u_n\} = \{v_1, \ldots, v_n\}$. Let $|G| = q_1^{m_1} \cdots q_t^{m_t}$ be the prime decomposition of the order of G. Finally, we define the following numbers:

$$d = d(|G|) = \text{g.c.d.}(q_1 - 1, \ldots, q_t - 1),$$

$$\delta = \delta(|G|) = \text{g.c.d.}(q_1^2 - 1, \ldots, q_t^2 - 1),$$

$$\mu_{q_i} = \mu_{q_i}(|G|) = (q_i - 1)/\text{g.c.d.}(|G|, q_i - 1),$$

$$\eta = \eta(n) = 1 \text{ or } 0 \text{ according to whether } n \text{ is an odd or even number,}$$

and

$$\delta^\pi_{|G|} = \text{g.c.d.}(p_1^2 - 1, \ldots, p_u^2 - 1, p_{u+1} - 1, \ldots, p_t - 1),$$

where $\pi(G) \cap \pi = \{p_1, \ldots, p_u\}$ and $\pi(G) = \{p_1, \ldots, p_u, p_{u+1}, \ldots, p_t\}$; in case $\pi(G) \cap \pi = \varnothing$ we define $\delta^\pi_{|G|} = d(|G|)$. Evidently, $\delta^{\pi(G)}_{|G|} = \delta(|G|)$.

When a finite group is analysed in a simple way from some of its subgroups, information about the number of conjugacy classes of G can frequently be obtained from a knowledge of properties of the numbers $r_G(gN)$, $g \in G$, by the use of some inductive principle to transfer the information to the whole group.

Specifically we shall prove the following results in Section 1:

(1.A) Let N and M be normal subgroups of G such that $N \leqq M$ and let $g \in G$. If $\{g\bar{m}_1, \ldots, g\bar{m}_t\}$ is a complete system of representatives from distinct conjugacy classes of $\bar{G} = G/N$ that make up the normal set $\bigcup_{x\in G} g^x\bar{M}$, then

$$t = r_G(\bar{g}\bar{M}), \quad r_G^\pi(gM) = \sum_{i=1}^{t} r_G^\pi(gm_iN) \quad \text{and} \quad {}_\pi\Delta_{gM}^G \sim ({}_\pi\Delta_{gm_1N}^G, \ldots, {}_\pi\Delta_{gm_tN}^G).$$

(1.B) Let N be a normal subgroup of G and let g be an element of G such that $o(g)$ and $|N|$ are relatively prime numbers. Suppose that $\bar{g}$ is a π-element of $\bar{G} = G/N$. Then we have

$$r_G^\pi(gN) = r_{C_G(g)}^\pi(C_N(g)) \quad \text{and} \quad {}_\pi\Delta_{gN}^G = {}_\pi\Delta_{C_N(g)}^{C_G(g)}.$$

(1.C) Let A be an abelian normal subgroup of G and let g be an element of G. Set

$$\tilde{A} = A/[g, C_{N_G(gA)}(A)] \quad \text{and} \quad \tilde{B} = N_G(gA)/[g, C_{N_G(gA)}(A)]$$

and suppose that $\tilde{g}$ is a π-element. Then $r_G^\pi(gA) = r_{\tilde{B}}^\pi(\tilde{g}\tilde{A})$ and ${}_\pi\Delta_{gA}^G = {}_\pi\Delta_{\tilde{g}\tilde{A}}^{\tilde{B}}$. Consequently the following inequality holds:

$$r_G(gA) \leqq |C_A(g)| / |[g, C_{N_G(gA)}(A)]/[g, A]|,$$

and the above bound is attained if and only if $\tilde{g}\tilde{A}$ is contained in $Z(\tilde{B})$. In particular, if $A \leqq Z(N_G(gA))$ we get $r_G^\pi(gA) = |A/[g, N_G(gA)]|_\pi$ (here, m_π denotes the π-part of the number m). Furthermore, if $|A| = p$, with p a prime number, then

$$r_G(gA) = \begin{cases} 1 & \text{if } g \notin Z(C_{N_G(gA)}(A)) \\ 1 + (p-1)/|N_G(gA)/C_{N_G(gA)}(A)| & \text{if } g \in Z(C_{N_G(gA)}(A)) \end{cases}$$

holds.

(1.A) is a reduction lemma; by using (1.A) one can analyse the numbers $r_G^\pi(xM)$ in terms of the numbers $r_G^\pi(yN)$. Frequently (1.A) and (1.B) are jointly used. Further, (1.B) allow us to get the conjugacy-vector $\Delta_{N \times_\lambda K}$ of a semidirect product $N \times_\lambda K$, in terms of the tuples Δ_N and Δ_K, and the action λ, whenever $|N|$ and $|K|$ are relatively prime numbers. In particular, here we shall establish a result which generalizes P. Hall's result (unpublished) (cf. [3] V.15.2) and we obtain the number of conjugacy classes in a finite supersoluble group, by considering an ordered Sylow Tower. (1.C) discusses new properties about the number $r_G^\pi(gA)$, which are used for classifying finite groups according to the number of conjugate classes (cf. [7]–[13]).

In Section 2 we prove the following congruences for each N normal subgroup of G:

(2.A) $r^\pi(G) \equiv |G/N| r^\pi(N) \pmod{\delta_{|G/N|}^\pi}$.

(2.B) $r^\pi(G) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|G|)\delta_{|G/N|}^\pi}$.

(2.C) If t divides $\delta^{\pi}_{|G/N|}$, then the following congruences are equivalent:

(i) $r^{\pi}(G) \equiv |G| \pmod t$,

(ii) $r^{\pi}(N) \equiv |N| \pmod t$.

(2.D) If t' divides $d(|G|)\delta_{|G/N|}$, then the following congruences are equivalent:

(i) $r(G) \equiv |G| \pmod{t'}$,

(ii) $r(N) \equiv |N| \pmod{t'}$;

in particular, if N is an abelian normal subgroup of G, then (2.D) yields

(2.E) $r(G) \equiv |G| \pmod{d(|G|)\delta_{|G/N|}}$,

and from (2.E) the following generalization follows easily:

(2.E′) $r(G) \equiv |G| \pmod{d(|G|)\delta_{|G:A|}}$,

whenever A is an abelian subnormal subgroup of G.

The above congruences are substantial improvements of known congruences of A. Vera-López and L. Ortíz de Elguea (cf. [12]) and A. Mann (cf. [4]). (2.E′) can also be obtained by arguing as A. Mann in [4] p. 83 and by using Itô's Theorem about the degrees of complex irreducible characters of a finite group. However, our proofs of all the above congruences are elementary, because no result of Character Theory is used.

On the other hand, the number of conjugacy classes of a supersoluble finite group has been investigated by M. Cartwright (cf. [1]), by showing the following inequality:

$$r(G) \geqq 0.6 \log_2 |G|.$$

The above logarithmic bound for supersoluble groups can be improved, arguing in a different way. Suppose that $|G| = \Pi_{i=1}^{t} p_i^{2n_i+e_i}$ is the factorization of the order of G in primary powers, being $n_i \geqq 0$ and $e_i = 0$ or 1 for all i, and suppose that G is supersoluble. Then we get the following results (into Section 3):

(3.A) $$r(G) \geqq \sum_{n_i \neq 0} 2^{\eta(|G|)} n_i p_i + \sum_{n_i = 0} (p_i - 1)^{1/2}.$$

(3.B) If $2 \nmid |G|$, then there exists a non-negative integer number k such that

$$r(G) = \sum_{i=1}^{t} (2n_i(p_i - 1) + e_i \mu_{p_i}) + 1 + k \cdot d(|G|)^2.$$

Evidently, (3.A) and (3.B) improve M. Cartwright's bound; indeed we substitute for the logarithmic function another, which increases more quickly.

On the other hand, if G is a supersoluble group and p is the smallest prime number dividing $|G|$, then there exists a non-negative number k such that

$$\text{(3.C)}\quad p^a \cdot r(G) = (p+1)(p^a - 1) + r(O_{p'}(G)) = k \cdot (p^2 - 1) \cdot d(|G|),$$

where p^a is the greatest power of p dividing $|G|$. In particular, if G has even order and $2^a \parallel |G|$ then we have

$$2^a \cdot r(G) = 3 \cdot (2^a - 1) + r(O_{2'}(G)) + k \cdot 3 \qquad \text{for some } k \geqq 0,$$

and the number $r(O_{2'}(G))$ can be estimated by using (3.B), since $O_{2'}(G)$ is a supersoluble group of odd order.

Let N be a normal subgroup of G. In Section 4 we analyse the number of conjugacy classes that make up the normal subgroup N, that is, $r_G(N)$.

We obtain the following results:

(4.A) There exists a non-negative number k such that

$$r^\pi(G) = r^\pi_G(N) + r^\pi(G/N) - 1 + k \cdot d(|G|) \cdot d(|G/N|).$$

The above equality is improved in case $\pi = \pi(G)$:

(4.B) There exists a non-negative number k such that

$$r(G) = r_G(N) + |N/(N \cap G')|(r(G/N) - 1) + k \cdot d(|G/N|)d(|G|).$$

In particular, when G is a finite p-group and putting $N = Z(G)$ or $\phi(G)$ in (4.B) ($\phi(G)$ denotes Frattini's subgroup of G), we get

$$\text{(4.B}')\quad r(G) = |Z(G)| + |Z(G)/(Z(G) \cap G')|(r(G/Z(G)) - 1) + k \cdot (p-1)^2,$$

$$\text{(4.B}'')\quad r(G) = r_G(\phi(G)) + |\phi(G)/G'|(|G/\phi(G)| - 1) + k \cdot (p-1)^2.$$

We also analyse the number $r(G)$ of conjugacy classes of a finite p-group G, in terms of the nilpotent class of G and of the cardinalities of terms of the upper central series, and we refine Sherman's inequality given in [6].

1. The number $r^\pi_G(gN)$

LEMMA (1.1). *Let N, M be two normal subgroups of G such that $N \leqq M$ and let g be an element of G. If $\{\bar{g}\bar{m}_1, \ldots, \bar{g}\bar{m}_t\}$ is a complete system of representatives from distinct conjugacy classes of $\bar{G} = G/N$ that make up the normal set $\bigcup_{x \in G} \bar{g}^x \bar{M}$, then*

$$t = r_{\bar{G}}(\bar{g}\bar{M}), \quad r^\pi_G(gM) = \sum_{i=1}^{t} r^\pi_G(gm_i N) \quad \text{and} \quad {}_\pi\Delta^G_{gM} \sim ({}_\pi\Delta^G_{gm_1N}, \ldots, {}_\pi\Delta^G_{gm_tN}).$$

PROOF. We claim that $[\bigcup_{x\in G} g^x M]_\pi = \dot{\bigcup}_{i=1}^{t}[\bigcup_{z\in G}(gm_i)^z N]_\pi$. Indeed, clearly $(gm_i)^z N$ is a subset of $g^z M$ for all $z \in G$ and each $i = 1, \ldots, t$, hence $\supseteq$ is evident. On the other hand, if $(gm)^x$ is a π-element, with $m \in M$, then there exists $y \in G$ such that $\overline{(gm)^x} = (\bar{g}\bar{m}_i)^y$ for some i, so that $(gm)^x$ is an element of $(gm_i)^y N$ and thus equality holds. Further, if there exist $x, y \in G$ and $n_1, n_2 \in N$ such that $(gm_i n_1)^x = (gm_j n_2)^y$, then $(\bar{g}\bar{m}_i)^{\bar{x}} = (\bar{g}\bar{m}_j)^{\bar{y}}$, that is, $\bar{g}\bar{m}_i$ is $\bar{G}$-conjugate to $\bar{g}\bar{m}_j$, and consequently $i = j$. Now, (1.1) follows immediately from the above disjoint union.

LEMMA (1.2). *Let N be a normal subgroup of G and let g be a π-element of G such that $o(g)$ and $|N|$ are relatively prime numbers. Then we have $r_G^\pi(gN) = r_{C_G(g)}^\pi(C_N(g))$ and ${}_\pi\Delta_{gN}^G = {}_\pi\Delta_{C_N(g)}^{C_G(g)}$.*

PROOF. Let us prove that $gC_N(g)$ contains a complete system of representatives from distinct conjugacy classes that intersect the coset gN. Let gn be an element of gN such that $o(gn) = o(g)$. By using the Schur–Zassenhaus Theorem (applied to $N\langle g\rangle$), $\langle gn\rangle$ is $N\langle g\rangle$-conjugate to $\langle g\rangle$, hence there exists $g^l n'$, with $n' \in N$ such that $gn = (g^j)^{g^l n'} = (g^j)^{n'}$ for some j coprime to $o(g) = o(\bar{g})$. Consequently $\bar{g} = \bar{g}^j$ and necessarily $j = 1$. Thus gn is G-conjugate to g and there exists just one conjugate class of elements of order $o(g)$ contained in $\bigcup_{x\in G} g^x N$. Suppose that $o(gn) \neq o(g)$. Clearly, $o(gn) = o(g)\cdot t$ for some t dividing $|N|$, and by using Bezout's Theorem, there exist integer numbers u and v such that $1 = u\cdot o(g) + v\cdot t$. Therefore $gn = (gn)^k \cdot m$ with $k = tv$ and $m = (gn)^{o(g)}$ lie in $C_N(gn)$. Further $o((gn)^t) = o(g)$ and g.c.d.$(k, o(g)) = 1$ imply $o((gn)^k) = o(g)$, so $(gn)^k = g^{x^{-1}}$ for some $x \in G$ and $(gn)^x = g\cdot m^x$, with $m^x \in (C_N((gn)^k))^x = C_N(g)$. Thus we have $r_G(gN) = r_G(gC_N(g))$.

On the other hand, for any $n_1, n_2 \in C_N(g)$, gn_1 and gn_2 are G-conjugates if and only if $gn_1 = g^x \cdot n_2^x$ for some $x \in G$, but then

$$g = (gn_1)_{\pi(o(g))} = (g^x \cdot n_2^x)_{\pi(o(g))} = g^x,$$

hence gn_1 is $C_G(g)$-conjugate to gn_2. Therefore

$$r_G(gC_N(g)) = r_{C_G(g)}(C_N(g)).$$

Furthermore for each $n \in C_N(g)$ we have

$$C_G(gn) = C_G(g) \cap C_G(n) = C_{C_G(g)}(n),$$

thus we conclude the desirable equalities.

COROLLARY (1.3). *Let N be a normal subgroup of G and let g be a*

π-element of G. Suppose that g.c.d.$(o(gN), |N|) = 1$. *Then the following equalities hold:*

$$r_{\bar{G}}^{\pi}(gN) = r_{C_G(g^{|N|})}^{\pi}(C_N(g^{|N|})) \quad \textit{and} \quad {}_{\pi}\Delta_{gN}^{G} = {}_{\pi}\Delta_{C_N(g^{|N|})}^{C_G(g^{|N|})}.$$

PROOF. Set $\bar{G} = G/N$. We have $o(\bar{g}) = |\langle g\rangle N/N| = o(g)/|\langle g\rangle \cap N|$, and by using Bezout's Theorem, there exist integer numbers u and v such that $1 = uo(\bar{g}) + v|N|$. Consequently v is coprime to $o(\bar{g})$ and we have $gN = (g^{|N|})^v N$, being g.c.d.$(v, o(\bar{g}^{|N|})) = 1$. Now from [12] Th. (3.12) we conclude

$${}_{\pi}\Delta_{gN}^{G} = {}_{\pi}\Delta_{(g^{|N|})^v N}^{G} = {}_{\pi}\Delta_{g^{|N|}N}^{G}.$$

On the other hand, $o(g)$ divides $|\langle g\rangle N| = o(\bar{g})|N|$, hence $(g^{|N|})^{o(\bar{g})} = 1$ and so $o(g^{|N|})$ divides $o(\bar{g})$, whence g.c.d.$(o(g^{|N|}), |N|) = 1$ and (1.3) follows directly from (1.2).

THEOREM (1.4). *Let G be a semidirect product of N by H with action λ, that is $G = N \times_{\lambda} H$. Suppose that* g.c.d.$(|N|, |H|) = 1$ *and let $\{h_1 = 1, h_2, \ldots, h_t\}$ be a complete system of representatives from distinct conjugacy classes of a π-element of H. Then the following affirmations hold:*

(1) $r^{\pi}(G) = \Sigma_{i=1}^{t} r_{C_G(h_i)}^{\pi}(C_N(h_i))$, *being* $C_G(h_i) = C_N(h_i) \times_{\lambda} C_H(h_i)$ *for each* $i = 1, \ldots, t$.

(2) ${}_{\pi}\Delta_G^G \sim ({}_{\pi}\Delta_{C_N(h_1)}^{C_G(h_1)}, \ldots, {}_{\pi}\Delta_{C_N(h_t)}^{C_G(h_t)})$.

PROOF. This result follows directly from (1.3).

EXAMPLES

(1) Let G be a supersoluble finite group. Then G possesses an ordered Sylow Tower, that is, there is a series $1 = G_0 < G_1 < \cdots < G_s = G$ of normal subgroups of G such that for each $i = 1, \ldots, s$, $P_i = G_i/G_{i-1}$ is isomorphic to a Sylow p_i-subgroup of G, where $p_1, \ldots, p_s$ are the distinct prime divisors of $|G|$ and $p_1 > p_2 > \cdots > p_s$ (cf. [3] p.715, VI.9.1). Now from Zassenhaus's Theorem there exist actions λ_i, $i = 1, \ldots, s-1$ such that

$$G = [\cdots[[[P_1 \times_{\lambda_1} P_2] \times_{\lambda_2} P_3] \times \cdots] \times_{\lambda_{s-1}} P_s,$$

and by using repeatedly Theorem (1.4) the tuple Δ_G can be obtained.

(2) Let us consider the following semidirect product:

$$G = N \times_{\lambda} H = [[C_3 \times C_3] \times_{\lambda} C_3] \times_{\lambda} (C_2 \times C_2) = \langle d_1, d_2, e\rangle \times_{\lambda} \langle b_1, b_2\rangle$$

with relations

$$\begin{cases} d_1^e = d_1 \\ d_2^e = d_1 d_2 \end{cases} \qquad \begin{cases} d_1^{b_1} = d_1^{-1} \\ d_2^{b_1} = d_2^{-1} \\ e^{b_1} = e \end{cases} \qquad \begin{cases} d_1^{b_2} = d_1^{-1} \\ d_2^{b_2} = d_2 \\ e^{b_2} = e^{-1} \end{cases}$$

Evidently, we have

$$C_G(b_1) = \langle e \rangle \langle b_1, b_2 \rangle, \quad C_G(b_2) = \langle d_2 \rangle \langle b_1, b_2 \rangle \quad \text{and} \quad C_G(b_1 b_2) = \langle d_1 \rangle \langle b_1, b_2 \rangle.$$

Therefore $\Delta^G_{b_1N} = \Delta^{C_G(b_1)}_{C_N(b_1)} = (12,6)$ and similarly $\Delta^G_{b_2N} = (12,6) = \Delta^G_{b_1b_2N}$. Also we have $\Delta^G_N = (108, 54, 18, 18, 9)$, therefore

$$r(G) = 11 \quad \text{and} \quad \Delta_G = (108, 54, 18, 18, 12, 12, 12, 9, 6, 6, 6)$$

(notice that it is easy to get the number of conjugacy classes that make up a normal subgroup, whenever the conjugacy relations are known!).

(3) Let p, q be two prime numbers satisfying $p \equiv 1 \pmod q$. Let p be the unique non-abelian p-group of order p^3 and exponent p, then

$$P = \langle a, b \mid a^p = 1 = b^p, [a, b] \text{ centralizes } a \text{ and } b \rangle.$$

Let t be such that $o(\bar{t}) = q$ in $((Z/pZ)*, \cdot) \simeq C_{p-1}$. Let us consider an automorphism α of P defined by $a^\alpha = a^t$ and $b^\alpha = b^{t^{-1}}$. Let $G = \mathrm{Hol}(P, \langle \alpha \rangle)$. Then

$$|G| = p^3 q \quad \text{and} \quad r(G) = r_G(P) + (q-1) r_G(\alpha P),$$

inasmuch as $r_G(\alpha^j P) = r_G(\alpha P)$ for all $j = 1, 2, \ldots, q-1$. Further,

$$r_G(\alpha P) = r_{C_P(\alpha)\langle \alpha \rangle}(C_P(\alpha)) \quad \text{and} \quad C_P(\alpha) = \langle [a, b] \rangle,$$

hence $r_G(\alpha P) = |C_P(\alpha)| = p$. But

$$r_G(P) = |Z(P)| + (|P| - |Z(P)|)/pq = p + (p^2 - 1)/q,$$

therefore

$$r(G) = p + (p^2 - 1)/q + (q-1)p = pq + (p^2-1)/q.$$

Finally

$$\begin{aligned} \Delta_G &\sim (\Delta^G_P, \Delta^G_{\alpha P}, \overset{p-1}{\ldots\ldots}, \Delta^G_{\alpha P}) \\ &\sim (p^3 q, \overset{p}{\ldots}, p^3 q, p^2, \overset{(p^2-1)/q}{\ldots\ldots\ldots}, p^2, pq, \overset{p(q-1)}{\ldots\ldots\ldots}, pq). \end{aligned}$$

Lemma (1.5). *Suppose that* $G = N \times_\lambda H$ *with* g.c.d.$(|N|, |H|) = 1$. *Let*

$\{n_1 = 1, n_2, \ldots, n_s\}$ be a complete system of representatives from distinct conjugacy H-classes of π-elements of N. Then we have

$$r^\pi(G) \leqq \sum_{j=1}^{s} r^\pi_{C_G(n_j)}(C_H(n_j)).$$

In particular, if N is abelian, the following equality holds:

$$r^\pi(G) = \sum_{j=1}^{s} r^\pi(C_H(n_j)). \tag{1}$$

PROOF. From Lemma (1.2) we deduce

$$r^\pi(G) = r^\pi_G\left(\bigcup_{h\in H} hC_N(h)\right). \tag{2}$$

In addition,

$$\left[\bigcup_{h\in H} hC_N(h)\right]_\pi = \left[\bigcup_{n\in N} C_H(n)n\right]_\pi = \left[\dot{\bigcup_{j=1}^{s}} \bigcup_{n\in \mathrm{Cl}_H(n_j)} C_H(n)n\right]_\pi.$$

Consequently (2) yields

$$r^\pi(G) \leqq \sum_{j=1}^{s} r^\pi_G(C_H(n_j)n_j).$$

Since $|N|$ and $|H|$ are relatively prime numbers it satisfies

$$r^\pi_G(C_H(n_j)n_j) = r^\pi_{C_G(n_j)}(C_H(n_j)) \qquad \text{for all } j,$$

thus we conclude the required inequality.

In particular, if N is an abelian group, we have $C_G(n_j) = N \times_\lambda C_H(n_j)$, being g.c.d.$(|N|, |C_H(n_j)|) = 1$, therefore $r^\pi_{N\times_\lambda C_H(n_j)}(C_H(n_j)) = r^\pi(C_H(n_j))$ and (1) follows immediately.

In the following, $O_p(G)$ and $O_{p'}(G)$ denote (respectively) the largest normal subgroup of G of p-power order and the largest normal subgroup of G of order coprime to p.

Let G be a group of order p^{2n+e}, with p a prime, n a non-negative integer, and $e = 0$ or 1. Then the number of conjugacy classes of elements of G is of the form

$$(n + k(p-1))(p^2-1) + p^e \tag{3}$$

with k a non-negative integer. In particular G has at least $n(p^2-1)+p^e$ classes. This was established using representation theory by Philip Hall [2] (see also [3]). Later, J. Poland (cf. [5]) and A. Mann (cf. [4]) have shown the above result without using character theory.

Next we generalize this result, when G is an extension of an abelian group by a p-group. In fact, from (1) and (3) there follows directly the following result:

THEOREM (1.6). *Let G be a finite group and let N be an abelian normal subgroup of G such that G/N is p-group. Let π be a set of prime numbers such that $p\in\pi$ and let $\{x_1,\dots,x_t\}$ be a complete system of representatives from distinct conjugacy G-classes of π-elements of $O_{p'}(N)$; then we have*

$$r^\pi(G)=\left(\sum_{i=1}^{t} n_i\right)(p^2-1)+\left(\sum_{i=1}^{t} p^{e_i}\right)+k(p^2-1)(p-1),$$

for some non-negative integer k and $p^{2n_i+e_i}=|C_P(x_i)|$, with $n_i\geqq 0$, $e_i=0$ or 1 for all $i=1,\dots,t$ and P a Sylow p-subgroup of G.

For each non-empty subset S of G we define

$$[g,S]=\{[g,x]=g^{-1}x^{-1}gx \mid x\in S\}.$$

LEMMA (1.7). *Let A be an abelian normal subgroup of G and let g be an element of G. Set*

$$\tilde{A}=A/[g,C_{N_G(gA)}(A)] \quad \text{and} \quad \tilde{B}=N_G(gA)/[g,C_{N_G(gA)}(A)].$$

Suppose that $\tilde{g}$ is a π-element. Then we have $r^\pi_G(gA)=r^\pi_{\tilde{B}}(\tilde{g}\tilde{A})$. Further, if $(\tilde{g}\tilde{n}_1,\dots,\tilde{g}\tilde{n}_t)$ is a complete system of representatives from distinct conjugacy classes of π-elements of $\tilde{B}$ that make up the normal set $[\bigcup_{\tilde{b}\in\tilde{B}}\tilde{g}^{\tilde{b}}\tilde{A}]_\pi$, chosen so that gn_i is a π-element for all i, then $\{gn_1,\dots,gn_t\}$ is a complete system of representatives from distinct conjugacy clases of π-elements of G that make up the normal set $[\bigcup_{z\in G} g^zA]_\pi$ and we have

$${}_\pi\Delta^G_{gA}={}_\pi\Delta^{\tilde{B}}_{\tilde{g}\tilde{A}}.$$

PROOF. It can easily be proved that

$$[g,C_{N_G(gA)}(A)]=\{[g,x] \mid x\in C_{N_G(gA)}(A)\}$$

is a normal subgroup of $N_G(gA)$ contained in A.

Let $\{\tilde{g}\tilde{n}_1,\dots,\tilde{g}\tilde{n}_t\}$ be a complete system of representatives from distinct conjugacy classes of π-elements of $\tilde{B}$ that intersect $[\tilde{g}\tilde{A}]_\pi$, chosen so that gn_i is a π-element for all i. Set $N=[g, C_{N_G(gA)}(A)]$. By using Lemma (1.1) we have

$$r_G^\pi(gA)=r^\pi_{N_G(gA)}(gA)=r^\pi_{N_G(gA)}(gn_1N)+\cdots+r^\pi_{N_G(gA)}(gn_tN),$$

where $t=r_{\tilde{B}}^\pi(\tilde{g}\tilde{A})$.

Furthermore, for every i we have

$$\begin{aligned} gn_iN=gn_i[g, C_{N_G(gA)}(A)] &= \{gn_ig^{-1}z^{-1}gz \mid z\in C_{N_G(gA)}(A)\} \\ &= \{z^{-1}n_i^{g^{-1}}gz \mid z\in C_{N_G(gA)}(A)\} \\ &= \mathrm{Cl}_{C_{N_G(gA)}(A)}(gn_i). \end{aligned}$$

Thus we conclude $r_{N_G(gA)}(gn_iN)=1$ and since gn_i is a π-element, we conclude $r^\pi_{N_G(gA)}(gn_iN)=1$. Consequently $r_G^\pi(gA)=t=r_{\tilde{B}}^\pi(\tilde{g}\tilde{A})$.

Finally we have

$$|N|=|gn_iN|=|gn_iN\cap \mathrm{Cl}_{N_G(gA)}(gn_i)|=|N|\,|C_{\tilde{B}}(\tilde{g}\tilde{n}_i)|/|C_{N_G(gA)}(gn_i)|,$$

that is,

$$|C_{N_G(gA)}(gn_i)|=|C_{\tilde{B}}(\tilde{g}\tilde{n}_i)| \qquad \text{for all } i=1,\ldots,t.$$

Thus we conclude the desired result.

COROLLARY (1.8). *Let A be an abelian normal subgroup of G and let $\bar{g}$ be a π-element of $\bar{G}=G/A$ such that $A\leqq Z(N_G(gA))$. Then the following equality holds*:

$$r_G^\pi(gA)=|A/[g, N_G(gA)]|_\pi.$$

PROOF. We have $g_{\pi'}\in A$ and $r_G^\pi(gA)=r_G^\pi(g_\pi A)=r_{\tilde{B}}^\pi(\tilde{g}_\pi\tilde{A})$, where $\tilde{A}=A/[g_\pi, N_G(gA)]$ and $\tilde{B}=N_G(gA)/[g_\pi, N_G(gA)]$, since $C_{N_G(g_\pi A)}(A)=C_{N_G(gA)}(A)=N_G(gA)$. Further, $\tilde{g}_\pi$ is an element of the center of $\tilde{B}$, therefore

$$r_G^\pi(gA)=|\tilde{A}|_\pi=|A/[g_\pi, N_G(gA)]|_\pi=|A/[g, N_G(gA)]|_\pi.$$

COROLLARY (1.9). *Let A be an abelian normal subgroup of G and let $\bar{g}$ be a π-element of $\bar{G}=G/A$. Then $|[g, C_{N_G(gA)}(A)]/[g,A]|$ divides $|C_A(g)|$ and the following inequality holds*:

$$r_G(gA)\leqq |C_A(g)|/|[g, C_{N_G(gA)}(A)]/[g,A]|.$$

Further, equality holds if and only if $\tilde{g}\tilde{A}$ is contained in the center of $\tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are defined in Lemma (1.7).

PROOF. Evidently $[g,A]$ and $[g, C_{N_G(gA)}(A)]$ are two normal subgroups of

$N_G(gA)$ satisfying $[g, A] \leqq [g, C_{N_G(gA)}(A)]$. Furthermore, $|A|/|[g, A]| = |C_A(g)|$ and by using Lemma (1.7) we have

$$r_G(gA) = r_{\tilde{B}}(\tilde{g}\tilde{A}) \leqq |\tilde{A}| = |A/[g, A]|/|[g, C_{N_G(gA)}(A)]/[g, A]|,$$

and equality holds if and only if $r_{\tilde{B}}(\tilde{g}\tilde{A}) = |\tilde{A}|$, and since $\tilde{g}\tilde{A}$ is a normal subset of $\tilde{B}$, $r_{\tilde{B}}(\tilde{g}\tilde{A}) = |\tilde{A}|$ is equivalent to: $\tilde{g}\tilde{A}$ is central in $\tilde{B}$.

The information obtained in Lemma (1.7) is more precise than that given in Lemma (2.10) of [8]. Indeed Lemma (1.7) implies

$$r_G(gA) = r_{N_G(gA)/[g,A]}(\tilde{g}A/[g, A]) \quad (= r_{N_G(gA)/[g,A]}(A/[g, A]),$$

in case $N_G(gA) = AF$ and $C_F(\bar{g}) = \overline{C_F(g)}$, where $\overline{C_F(g)}$ is the image of $C_F(g)$ in $N_G(gA)/A$ and F is a subgroup of $N_G(gA)$). In addition, if A is a p-group, it follows that either $r_G^{\pi}(gA) = 1$ or $r_G^{\pi}(gA)$ is a power of p, assuming that $p \in \pi$. Furthermore, assume that G is a p-group and set $A = Z(G)$ and $\pi = \pi(G)$. Then for each $g \in G$, Corollary (1.8) yields

$$\begin{aligned} r_G(gZ(G)) &= |Z(G)/[g, N_G(gZ(G))]| \\ &= |C_G(g)|/|N_G(gZ(G))/Z(G)| \\ &= |C_G(g)|/|C_{\bar{G}}(\bar{g})|, \qquad \text{where } \bar{G} = G/Z(G). \end{aligned}$$

LEMMA (1.10). *Let A be a normal subgroup of G and assume that $A = C_p$, with p prime. Then the following affirmations hold*:

(1) *If $g \notin Z(C_{N_G(gA)}(A))$, then $r_G(gA) = 1$.*

(2) *If $g \in Z(C_{N_G(gA)}(A))$, then we have*

$$r_G(gA) = 1 + (p-1)/|N_G(gA)/C_{N_G(gA)}(A)|.$$

PROOF. If $g \notin Z(C_{N_G(gA)}(A))$ we have $[g, C_{N_G(gA)}(A)] = A$ and $r_G(gA) = r_{\tilde{B}}(\tilde{g}\tilde{A}) = 1$. Let us suppose that g centralizes $C_{N_G(gA)}(A)$ and set

$$N_G(gA) = \dot{\bigcup_{j=1}^{u}} y_j C_{N_G(gA)}(A).$$

Then we have

$$\begin{aligned} r_G(gA) &= (1/|N_G(gA)|)|T_{gA, N_G(gA)}| \\ &= (1/|N_G(gA)|) \sum_{j=1}^{u} |T_{gA, y_j C_{N_G(gA)}(A)}|. \end{aligned}$$

We now consider two cases.

(a) If $N_G(gA) = C_{N_G(gA)}(A)$, the $g \in Z(N_G(gA))$ and so $r_G(gA) = p$.

(b) If $N_G(gA) \neq C_{N_G(gA)}(A)$, then for each $y_j \notin C_{N_G(gA)}(A)$ we have $r_G(y_jA) = 1$, because $|C_A(y_j)| = 1$, hence $N_G(y_jA) = C_G(y_j)A$. Since $g \in N_G(y_jA)$, it follows that there exists $a_0 \in A$ such that $[ga_0, y_j] = 1$. Further, ga_0 is the unique element of the coset gA commuting with y_j (otherwise, $[y_j, A] = 1$). Thus, we have

$$|T_{gA, y_j C_{N_G(gA)}(A)}| = |C_{N_G(gA)}(A)| \qquad \text{for each } j \neq 1,$$

$$|T_{gA, C_{N_G(gA)}(A)}| = |A|\,|C_{N_G(gA)}(A)| \qquad \text{for } j = 1.$$

Finally we conclude

$$\begin{aligned} r_G(gA) &= (1/|N_G(gA)|)((|N_G(gA)/C_{N_G(gA)}(A)| - 1)|C_{N_G(gA)}(A)| + |A|\,|C_{N_G(gA)}(A)|) \\ &= 1 + (|A| - 1)/|N_G(gA)/C_{N_G(gA)}(A)|. \end{aligned}$$

Remarks

(1) Let N be a normal subgroup of G and let us consider the application

$$i : G/C_G(N) \to \mathrm{Aut}(N)$$

defined by

$$\bar{g} \mapsto i(\bar{g}) : n \mapsto n^g = g^{-1}ng$$

for all $n \in N$. Then

$$r_G(N) = r_{N \times_i G/C_G(N)}(N) = r_{\mathrm{Hol}(N, G/C_G(N))}(N).$$

In particular, it is

$$r_G(N) \geqq r_{\mathrm{Hol}(N)}(N).$$

Indeed, if $n_1^g = n_2$ for some $g \in G$, then $\bar{n}_1^{\bar{g}} = \bar{n}_2$ in $\bar{N} \times_i \bar{G}$, being $\bar{G} = G/C_G(N)$ and $\bar{N} \simeq N$. Conversely, if $\bar{n}_1$ is $\bar{N} \times_i \bar{G}$-conjugate to $\bar{n}_2$ then there exists $\bar{n}i(\bar{g})$ so that $\bar{n}_1^{\bar{n}i(\bar{g})} = \bar{n}_2$, that is, $(n_1^n)i(\bar{g}) = n_2$, and consequently n_1 is G-conjugate to n_2.

For example, if $N = \mathrm{Hol}(C_p, C_q)$ is a normal subgroup of G, we have

$$r_G(N) = r_{\mathrm{Hol}(N, G/C_G(N))}(N) = 1 + ((p-1)/q)/|G/C_G(N)|_{p'} + q - 1,$$

where $|G/C_G(N)|_{p'}$ denotes the p'-part of the number $|G/C_G(N)|$ (this follows immediately from the structure of $\mathrm{Hol}(C_p) = C_p \times_f C_{p-1}$).

(2) Let N be a normal subgroup of G such that $Z(N) = 1$. Then we have

$$r_G(N) = r_{G/C_G(N)}(NC_G(N)/C_G(N)).$$

(This result can be shown easily.)

2. Conjugacy classes of π-elements

In the following, s_g^π denotes the number of conjugacy N-classes of a π-element of N fixed by the automorphism $f_g : N \to N$ defined by $f_g(x) = x^g$ for all $x \in N$.

LEMMA (2.1). *Let N be a normal subgroup of G and let g be an element of G. Set $\bar{G} = G/N$. Suppose that $\bar{g}$ is a π-element. Then we have $s_g^\pi = r^\pi_{N\langle g\rangle}(gN) = s^\pi_{g^j}$ for each j coprime to $o(\bar{g})$.*

PROOF. Cf. [12] Theorem (3.2) Remark (1).

LEMMA (2.2). *Let N be a normal subgroup of G such that G/N is soluble. Then the following congruences hold*:

(i) $r^\pi(G) \equiv |G/N| \cdot r^\pi(N) \pmod{\delta^\pi_{|G/N|}}$.

(ii) $r^\pi(G) \equiv r^\pi(G/N) \cdot r^\pi(N) \pmod{d(|G|)\delta^\pi_{|G/N|}}$.

PROOF. Arguing by induction on $|G/N|$, we shall prove both (i) and (ii). Clearly, the result is true, in case $G/N = 1$. Let us suppose that $G/N = \langle \bar{g} \rangle \simeq C_p$, with p prime. We have

$$r^\pi(G) = r^\pi_G(N) + (p-1)r^\pi_G(gN)$$

and also

$$r^\pi_G(N) = (r^\pi(N) + (p-1)s_g^\pi)/p.$$

If $\bar{g}$ is not a π-element (hence $p \notin \pi$), then $r^\pi_G(gN) = 0$ and

$$pr^\pi(G) = r^\pi(N) + (p-1)s_g^\pi. \tag{4}$$

Since $p \equiv 1 \pmod{\delta^\pi_{|G/N|}}$, from (4) it follows that

$$r^\pi(G) \equiv r^\pi(N) \equiv |G/N|\, r^\pi(N) \pmod{\delta^\pi_{|G/N|}},$$

and (i) is proved.

On the other hand, we know that $s_g^\pi \equiv 1 \pmod{d(|G|)}$ and $pr^\pi(G)$ is congruent to $r^\pi(G) + (p-1)$ modulo $d(|G|)\delta^\pi_{|G/N|}$, inasmuch as $r^\pi(G) \equiv 1 \pmod{d(|G|)}$, thus (4) yields

$$r^\pi(G) + (p-1) \equiv r^\pi(N) + (p-1)s_g^\pi \equiv r^\pi(N) + (p-1) \pmod{d(|G|)\delta^\pi_{|G/N|}}$$

and consequently

$$r^\pi(G) \equiv r^\pi(G/N) \cdot r^\pi(N) \pmod{d(|G|)\delta^\pi_{|G/N|}},$$

because $r^\pi(G/N) = 1$.

If $\bar{g}$ is a π-element (that is, $p \in \pi$), then $r^\pi_G(gN) = s^\pi_g$ and we have

$$pr^\pi(G) = (p^2 - 1)s^\pi_g + r^\pi(N)$$

hence

$$p^2 \cdot r^\pi(G) = p(p^2 - 1)s^\pi_g + pr^\pi(N). \tag{5}$$

Since p^2 is congruent to 1 modulo $\delta^\pi_{|G/N|}$ we have

$$p^2 r^\pi(G) \equiv r^\pi(G) + p^2 - 1 \pmod{d(|G|)\delta^\pi_{|G/N|}}$$

and also

$$p(p^2 - 1)s^\pi_g \equiv p^2 - 1 \pmod{d(|G|)\delta^\pi_{|G/N|}},$$

hence the following assertions follow from (5):

$$r^\pi(G) \equiv pr^\pi(N) = |G/N| r^\pi(N) = r^\pi(G/N) r^\pi(N) \pmod{d(|G|)\delta^\pi_{|G/N|}}.$$

We now suppose the lemma true for each group G_1 and each $N_1 \trianglelefteq G_1$ such that G_1/N_1 is soluble and $|G_1/N_1| < |G/N|$. Let $1 \neq L/N \trianglelefteq G/N$ such that $(G/N)/(L/N) \simeq C_{p_1}$, with p_1 a prime number. Then G/L is isomorphic to C_{p_1} and arguing as above we have

$$\begin{aligned} r^\pi(G) &\equiv |G/L| r^\pi(L) \pmod{\delta^\pi_{|G/L|}}, \\ r^\pi(G) &\equiv r^\pi(G/L) r^\pi(L) \pmod{d(|G|)\delta^\pi_{|G/L|}}. \end{aligned} \tag{6}$$

On the other hand, applying the inductive hypothesis to pair (L, N) yields

$$\begin{aligned} r^\pi(L) &\equiv |L/N| r^\pi(N) \pmod{\delta^\pi_{|L/N|}}, \\ r^\pi(L) &\equiv r^\pi(L/N) r^\pi(N) \pmod{d(|L|)\delta^\pi_{|L/N|}}, \end{aligned} \tag{7}$$

and applying it to the pair $(G/N, L/N)$ we have

$$r^\pi(G/N) \equiv r^\pi(G/L) r^\pi(L/N) \pmod{d(|G/N|)\delta^\pi_{|G/L|}}. \tag{8}$$

Consequently, from (6), (7) and (8) we get

$$r^\pi(G) \equiv |G/L|\,|L/N| r^\pi(N) = |G/N| r^\pi(N) \pmod{\delta^\pi_{|G/N|}}$$

and

$$r^\pi(G) \equiv r^\pi(G/L) r^\pi(L/N) r^\pi(N) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|G|)\delta^\pi_{|G/N|}},$$

because $d(|G|)$ (resp. $\delta^{\pi}_{|G/N|}$) divides both $d(|G/N|)$ and $d(|L|)$ (resp. $\delta^{\pi}_{|G/L|}$ and $\delta^{\pi}_{|L/N|}$). Thus the lemma is proved.

THEOREM (2.3). *Let N be a normal subgroup of G. Then the following congruences hold*:

(i) $r^{\pi}(G) \equiv |G/N| r^{\pi}(N) \pmod{\delta^{\pi}_{|G/N|}}$.

(ii) $r^{\pi}(G) \equiv r^{\pi}(G/N) r^{\pi}(N) \pmod{d(|G|)\delta^{\pi}_{|G/N|}}$.

PROOF. We have $|G/N|^2 \equiv 1 \pmod{\delta^{\pi}_{|G/N}}$, hence (i) is equivalent to showing

$$|G/N| r^{\pi}(G) \equiv r^{\pi}(N) \pmod{\delta^{\pi}_{|G/N|}},$$

and this is equivalent to

$$|G| r^{\pi}(G) \equiv |N| r^{\pi}(N) \pmod{|N| \delta^{\pi}_{|G/N|}},$$

that is,

$$|T_{[G]_{\pi},G}| \equiv |T_{[N]_{\pi},N}| \pmod{|N| \delta^{\pi}_{|G/N|}}. \tag{9}$$

We shall prove (9). Set

$$\Omega = \{(x, y) \in [G]_{\pi} \times G \mid [x, y] = 1 \text{ and } x \text{ or } y \text{ is not an element of } N\}.$$

We have $|T_{[G]_{\pi},G}| = |T_{[N]_{\pi},N}| + |\Omega|$, hence it is sufficient to show the following congruence;

$$|\Omega| \equiv 0 \pmod{|N| \delta^{\pi}_{|G/N|}}. \tag{10}$$

Set $\mathrm{T} = \{\langle x, y \rangle \mid (x, y) \in \Omega\}$. We have the following decomposition:

$$\Omega = \bigcup_{H \in \mathrm{T}} (T_{[NH]_{\pi},NH} - T_{[N]_{\pi},N}).$$

If H_1, H_2 belong to T, then we have

$$(T_{[NH_1]_{\pi},NH_1} - T_{[N]_{\pi},N}) \cap (T_{[NH_2]_{\pi},NH_2} - T_{[N]_{\pi},N}) = T_{[NH_1 \cap NH_2]_{\pi},NH_1 \cap NH_2} - T_{[N]_{\pi},N}$$

and $(NH_1 \cap NH_2)/N$ is abelian. Therefore there exist subgroups $H_1, \ldots, H_u$ of G such that $N < H_i$, H_i/N is abelian and

$$|\Omega| = \sum_{i=1}^{u} (-1)^{r_i} |T_{[H_i]_{\pi},H_i} - T_{[N]_{\pi},N}|, \tag{11}$$

for some natural numbers r_i, $i = 1, \ldots, u$.

It follows from Lemma (2.2) that $r^{\pi}(H_i)$ is congruent to $|H_i/N| r^{\pi}(N)$ modulo $\delta^{\pi}_{|H_i/N|}$, or equivalently

$$|T_{[H_i]_\pi, H_i}| \equiv |T_{[N]_\pi, N}| \pmod{|N| \delta^\pi_{|H_i/N|}},$$

but $\delta^\pi_{|G/N|}$ divides $\delta^\pi_{|H_i/N|}$ hence (10) follows directly from (11).

To prove (ii) we consider $\Gamma = \{H \leqq G \mid HN/N \text{ is abelian}\}$. Evidently the following decompositions are satisfied:

$$T_{[G]_\pi, G} = \bigcup_{H \in \Gamma} T_{[NH]_\pi, NH}$$

and

$$T_{[G/N]_\pi, G/N} = \bigcup_{H \in \Gamma} T_{[HN/N]_\pi, HN/N}.$$

Again, there exist subgroups $K_1, \ldots, K_v$ of G such that K_i/N is abelian for every i, and there exist natural numbers $r_1, \ldots, r_v$ so that

$$|T_{[G]_\pi, G}| = \sum_{i=1}^{v} (-1)^{r_i} |T_{[K_i]_\pi, K_i}|$$

(12) and

$$|T_{[G/N]_\pi, G/N}| = \sum_{i=1}^{v} |T_{[K_i/N]_\pi, K_i/N}| (-1)^{r_i}.$$

From Lemma (2.2)

$$r^\pi(K_i) \equiv r^\pi(K_i/N) r^\pi(N) \pmod{d(|K_i|) \delta^\pi_{|K_i/N|}},$$

hence

$$|T_{[K_i]_\pi, K_i}| \equiv |T_{[K_i/N]_\pi, K_i/N}|\, |T_{[N]_\pi, N}| \pmod{|N| d(|G|) \delta^\pi_{|G/N|}} \tag{13}$$

since $d(|G|)$ divides $d(|K_i|)$ and $\delta^\pi_{|G/N|}$ divides $\delta^\pi_{|K_i/N|}$ for $K_i/N \neq 1$ (in case $K_i/N = 1$, (13) is trivial). Now, from (12) and (13) we get

$$|T_{[G]_\pi, G}| \equiv \sum_{i=1}^{v} (-1)^{r_i} |T_{[K_i/N]_\pi, K_i/N}|\, |T_{[N]_\pi, N}| = |T_{[N]_\pi, N}|\, |T_{[G/N]_\pi, G/N}|$$

modulo $|N| d(|G|) \delta^\pi_{|G/N|}$, that is,

$$|G| r^\pi(G) \equiv |G/N| r^\pi(G/N) |N| r^\pi(N) \pmod{|N| d(|G|) \delta^\pi_{|G/N|}},$$

and consequently

$$r^\pi(G) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|G|) \delta^\pi_{|G/N|}},$$

because $|G/N|$ is coprime to both $\delta^\pi_{|G/N|}$ and $d(|G|)$.

COROLLARY (2.4). *The following congruence holds*:

$$(14) \qquad r^{\pi}(G) \equiv |G| \pmod{\delta^{\pi}_{|G|}}.$$

PROOF. This result follows directly from Theorem (2.3) (part (i)) putting $N = 1$.

Thus (2.3) generalizes (14). This last congruence was shown in [12] (Th. (3.16)). Further, the congruence

$$r^{\pi}(G) \equiv |G| \pmod{d(|G|)\delta^{\pi}_{|G|}}$$

is not generally true. Indeed, let us consider $G = (C_5 \times C_5) \times_f C_3$, the Frobenius's group of kernel $C_5 \times C_5$ and complement isomorphic to C_3 and let $\pi = \{5\}$. Then we have

$$r^{\pi}(G) = 1 + (5^2 - 1)/3 = 9, \quad \delta^{\pi}_{|G|} = 2 = d(|G|)$$

and

$$|G| - r^{\pi}(G) = 66 \not\equiv 0 \pmod 4.$$

However, (2.A) and (2.B) allow us to get the following criteria, which connect the arithmetical structures of the numbers $r(N) - |N|$ and $r(G) - |G|$.

COROLLARY (2.5). *Let N be a normal subgroup of G and let t be a number dividing $\delta^{\pi}_{|G/N|}$. Then the following affirmations are equivalent*:

(1) $r^{\pi}(G) \equiv |G| \pmod t$.

(2) $r^{\pi}(N) \equiv |N| \pmod t$.

PROOF. It follows immediately from Theorem (2.3) (part (i)).

COROLLARY (2.6). *Let N be a normal subgroup of G and let t be a number dividing $d(|G|)\delta_{|G/N|}$. Then the following affirmations are equivalent*:

(1) $r(G) \equiv |G| \pmod t$.

(2) $r(N) \equiv |N| \pmod t$.

In particular, if N is abelian, the following congruence holds:

$$(15) \qquad r(G) \equiv |G| \pmod{d(|G|)\delta_{|G/N|}}.$$

PROOF. It follows immediately from Theorem (2.3) (part (ii)).

COROLLARY (2.7). *If A is an abelian subnormal subgroup of G, then the following congruence holds*:

$$(16) \qquad r(G) \equiv |G| \pmod{d(|G|)\delta_{|G:A|}}.$$

PROOF. Obviously, if $\pi(|A|) \subseteq \pi(|G:A|)$ then we have $\delta_{|G:A|} = \delta_{|G|}$ and the result is true (here, $\pi(m)$ denotes the set of all prime numbers dividing m). Otherwise, we consider the different prime numbers $p_1, \ldots, p_s$ dividing $|A|$ and not contained in $\pi(|G:A|)$. If P_i is a Sylow p_i-subgroup of A, the conditions $P_i \operatorname{char} A \trianglelefteq\trianglelefteq G$ and $p_i \nmid |G:A|$ imply P_i normal in G, therefore $N = P_1 \cdots P_s = P_1 \times \cdots \times P_s$ is an abelian normal subgroup of G satisfying $\delta_{|G/N|} = \delta_{|G:A|}$ and consequently (16) follows directly from (15).

Obviously (16) improves the following (A. Mann's) congruence:

$$r(G) \equiv |G| \pmod{d(|G|)\delta_{|G|}}.$$

Examples

(1) Let us consider $G = \mathrm{Hol}(C_2 \times C_2 \times C_2, C_7)$. Then (15) yields $r(G) \equiv 8 \cdot 7 \equiv 8 \pmod{48}$ (indeed $r(G) = 8$), whereas A. Mann's congruence yields $r(G) \equiv 2 \pmod 3$.

(2) Consider $G = \mathrm{Hol}(C_2 \times C_2 \times C_2, C_7 \times_f C_3)$, then (15) yields $r(G) \equiv 8 \cdot 7 \cdot 3 \equiv 0 \pmod 8$. Indeed $r(G) = 8$, whereas A. Mann's congruence does not yield any information in this case.

Let us consider the following sets of prime numbers dividing $|G|$:

$$\pi = \{ p \in \pi(G) \mid v_p(|G|) \text{ is even}\},$$

$$\sigma = \{ p \in \pi(G) \mid v_p(|G/Z(G)|) \text{ is even}\},$$

$$\tau = \{ p \in \pi(G) \mid v_p(|G'|) \text{ is even}\}$$

(G' being the derived subgroup of G). Evidently we have the following congruences:

$$|G| \equiv 1 \pmod{\delta^{\pi}_{|G|}},$$

$$|G/Z(G)| \equiv 1 \pmod{\delta^{\sigma}_{|G|}},$$

$$|G'| \equiv 1 \pmod{\delta^{\tau}_{|G|}}.$$

Further $r(G)$ is congruent to $|G|$ modulo $\delta_{|G|}$ and the numbers $\delta^{\pi}_{|G|}, \delta^{\sigma}_{|G|}, \delta^{\tau}_{|G|}$ are divisors of $\delta_{|G|}$, therefore by observing that $r(G) \geqq |Z(G)|$ and $r(G) \geqq |G/G'|$ we have the following:

Corollary (2.8). *For each finite group G, there exist non-negative integer numbers k_i, $i = 1, 2, 3$, such that*

$$r(G) = 1 + k_1\delta^{\pi}_{|G|} = |Z(G)| + k_2\delta^{\sigma}_{|G|} = |G/G'| + k_3\delta^{\tau}_{|G|}.$$

Remark. Arguing as above, if there exists $N \trianglelefteq G$ such that $r(G) \geqq |N|$, then

$$r(G) = |N| + k \cdot \delta^{\rho}_{|G|} \qquad \text{for some } k \geqq 0,$$

where $\rho = \{ p \in \pi(G) \mid v_p(|G/N|) \text{ is even}\}$. Similarly, if $r(G) \geqq |G/N|$, then we have

$$r(G) = |G/N| + k' \cdot \delta^{\nu}_{|G|} \qquad \text{for some } k' \geqq 0$$

where $\nu = \{ p \in \pi(G) \mid v_p(|N|) \text{ is even}\}$. In general, if $q_1^{a_1}, \ldots, q_s^{a_s}$ are prime powers dividing $|G|$ such that $r(G) \geqq \prod_{i=1}^{s} q_i^{a_i}$, then there exists a non-negative integer number k'' such that

$$r(G) = \prod_{i=1}^{s} q_i^{a_i} + k'' \cdot \delta^{\partial}_{|G|},$$

where $\partial = \{ p \in \pi(G) - \{q_1, \ldots, q_s\} \mid v_p(|G|) \text{ is even}\}$.

Examples

(1) If $|G| = m^2$, then we have $r(G) = 1 + k \cdot \delta_{|G|}$ for some $k \geqq 0$.

(2) If $|G| = 2^{2n} \cdot 7^m$, then we have $r(G) = 1 + k \cdot 3$ for some $k \geqq 0$, setting $\pi = \{2\}$.

(3) For each finite group $G \neq 1$, the following inequality holds:

$$r(G) \geqq 1 + \delta^{\pi}_{|G|} + d(|G|)|\pi(G) - \pi|,$$

where $\pi = \{ p \in \pi(G) \mid v_p(|G|) \text{ is even}\}$. (Indeed, arguing as A. Mann in [4] p. 83, there exists $\ell = \ell(|G|)$ such that ℓ has exactly order $d(|G|)$ modulo any divisor ($\neq 1$) of $|G|$ and consequently g^{ℓ^i} is not G-conjugate to g^{ℓ^j} for all $i \neq j$, $1 \leqq i, j \leqq d(|G|)$.) Further, for each $q \in \pi(G)$, there are elements of order q in G. Thus the above inequality follows directly from (14). The above considerations also yield

$$r(G) \geqq 1 + d(|G|)|\pi(G)|.$$

Next, we analyse the arithmetical structure of the number $r^{\pi}(G) - r^{\pi}(G/N)r^{\pi}(N)$.

Lemma (2.9). *Let N be a normal subgroup of G and let g be an element of G. Then the following assertions are true*:

(i) $r^{\pi}(N) \equiv s_g^{\pi} \equiv 1 \pmod{d(|N|)}$.

(ii) $r_G^{\pi}(N) \equiv r^{\pi}(N) \pmod{d(|G/N|) \cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}$.

Proof. (i) cf. [12].

(ii) We have $s_g^\pi = (1/|N|)\,\Sigma_{n\in[N]_\pi} |C_G(n) \cap gN|$, since $|C_G(n)\cap gN| \neq 0$ if and only if $\mathrm{Cl}_N(n)^g = \mathrm{Cl}_N(n)$, where $|C_G(n)\cap gN| = |C_N(n)|$, in this case. Consequently the following equalities hold:

$$\begin{aligned} r_G^\pi(N) &= (1/|G|) \sum_{n\in[N]_\pi} |C_G(n)| \\ &= (1/|G|) \sum_{x\in G/N} \sum_{n\in[N]_\pi} |C_G(n)\cap xN| \\ &= (1/|G/N|) \sum_{x\in G/N} s_x^\pi, \end{aligned}$$

and so

$$|G/N|\, r_G^\pi(N) = \sum_{x\in G/N} s_x^\pi. \tag{17}$$

On the other hand, there exists a natural number $\ell = \ell(|G/N|)$ such that ℓ has exactly order $d(|G/N|)$ modulo any divisor ($\neq 1$) of $|G/N|$ and the following decomposition may be considered:

$$G/N = \{\bar{1}\} \cup \dot{\bigcup_{j=1}^{w}} \dot{\bigcup_{i=1}^{d(|G/N|)}} \{\bar{x}_j^{\ell^i}\}, \tag{18}$$

where $|G/N| = 1 + w\cdot d(|G/N|)$. Further we have $s_y^\pi = s_{x_j}^\pi$ for each element y contained in $\{x_j^{\ell^i} \mid 1 \leqq i \leqq d(|G/N|)\}$ and for all j, inasmuch as ℓ^i is coprime to $o(\bar{x}_j)$. Therefore from (17) and (18) we get

$$|G/N|\, r_G^\pi(N) = r^\pi(N) + \sum_{j=1}^{w} d(|G/N|) s_{x_j}^\pi,$$

but $s_{x_j}^\pi \equiv r^\pi(N) \pmod{d(|N|)}$, hence

$$d(|G/N|) s_{x_j}^\pi \equiv d(|G/N|) r^\pi(N) \pmod{d(|G/N|)d(|N|)}$$

and

$$\begin{aligned} |G/N|\, r_G^\pi(N) &\equiv (1 + wd(|G/N|)) r^\pi(N) \\ &= r^\pi(N)|G/N| \pmod{d(|G/N|)d(|N|)}, \end{aligned}$$

whence we conclude the desired congruence.

LEMMA (2.10). *Suppose that $G = N \times_\lambda H$ with* g.c.d.$(|N|, |H|) = 1$. *Then the following congruence holds*:

$$r^\pi(G) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}.$$

PROOF. Arguing as above, it may be considered a complete system of representatives from distinct conjugacy H-classes of π-elements of H of the following type:

$$\{1\} \cup \{h_j^{f^i} \mid 1 \leqq j \leqq w,\ 1 \leqq i \leqq d(|G/N|)\},$$

where $r^\pi(H) = 1 + w \cdot d(|G/N|)$. Now, from Theorem (1.4) (part (1)) we have

$$r^\pi(G) = r_G^\pi(N) + d(|G/N|) \sum_{j=1}^{w} r^\pi_{C_G(h_j)}(C_N(h_j)). \tag{19}$$

Since $C_N(h_j)$ is a normal subgroup of $C_G(h_j)$, from Lemma (2.9) (part (ii)) we deduce

$$\begin{aligned} r^\pi_{C_G(h_j)}(C_N(h_j)) &\equiv r^\pi(C_N(h_j)) \\ &\equiv 1 \pmod{d(|C_N(h_j)|)/\text{g.c.d.}(|C_G(h_j)|, d(|C_N(h_j)|))}. \end{aligned}$$

But clearly, $d(|N|)/\text{g.c.d.}(|G|, d(|N|))$ divides

$$d(|C_N(h_j)|)/\text{g.c.d.}(|C_G(h_j)|, d(|C_N(h_j)|))$$

in case $|C_N(h_j)| \neq 1$, hence we have

$$\begin{cases} r^\pi_{C_G(h_j)}(C_N(h_j)) \equiv r^\pi(N) \pmod{d(|N|)/\text{g.c.d.}(|G|, d(|N|))} \\ \text{and} \\ r^\pi_G(N) \equiv r^\pi(N) \pmod{d(|G/N|) \cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}. \end{cases} \tag{20}$$

Finally, from (19) and (20) we get

$$\begin{aligned} r^\pi(G) &\equiv r^\pi(N)(1 + wd(|G/N|)) \\ &= r^\pi(N) r^\pi(G/N) \pmod{d(|G/N|) \cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}, \end{aligned}$$

the required congruence.

LEMMA (2.11). *Let N be a normal subgroup of G such that G/N is abelian. Then the following congruence holds*:

$$r^\pi(G) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|G/N)| \cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}.$$

PROOF. We prove the lemma by induction on $|G/N|$. Let us assume $G/N = \langle \bar{g} \rangle \simeq C_p$, with p prime. If $p \notin \pi$, then $r^\pi(G/N) = 1$ and also

$$r^{\pi}(G) = r^{\pi}_G(N) \equiv r^{\pi}(N)$$
$$= r^{\pi}(G/N)r^{\pi}(N) \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))},$$

by using Lemma (2.9) (part (ii)). On the other hand, if $p \in \pi$, then we have

$$r^{\pi}(G) = r^{\pi}_G(N) + (p-1)s^{\pi}_g,$$

and again by using Lemma (2.9) we get

$$r^{\pi}(G) \equiv r^{\pi}(N) + (p-1)r^{\pi}(N)$$
$$= r^{\pi}(N)r^{\pi}(G/N) \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}$$

because $(p-1)s^{\pi}_g \equiv (p-1)r^{\pi}(N) \pmod{d(|G/N|)\cdot d(|N|)}$.

Thus we may assume true the lemma for every pair (G_1, N_1) satisfying the following conditions: $N_1 \trianglelefteq G_1$, G_1/N_1 abelian and $|G_1/N_1| < |G/N|$. Furthermore we may assume that there exists a prime number q dividing both $|N|$ and $|G/N|$, otherwise, by applying the Schur–Zassenhaus Theorem, we have $G = N \times_{\lambda} H$ for some subgroup H and (2.11) follows immediately from (2.10). Now, since G/N is abelian, there exists $L \trianglelefteq G$ such that $L/N \simeq C_q$ and, arguing as above for (L, N), we have

$$\text{(21)} \quad r^{\pi}(L) \equiv r^{\pi}(L/N)r^{\pi}(N) \pmod{d(|L/N|)\cdot d(|N|)/\text{g.c.d.}(|L|, d(|N|))}$$

Applying the induction hypothesis to both pairs (G, L) and $(G/N, L/N)$, we have the following congruences:

$$\text{(22)} \quad \begin{cases} r^{\pi}(G) \equiv r^{\pi}(G/L)r^{\pi}(L) \pmod{d(|G/L|)\cdot d(|L|)/\text{g.c.d.}(|G|, d(|L|))}, \\ r^{\pi}(G/N) \equiv r^{\pi}(G/L)r^{\pi}(L/N) \\ \qquad \pmod{d(|G/L|)\cdot d(|L/N|)/\text{g.c.d.}(|G/N|, d(|L/N|))}. \end{cases}$$

From (21) and (22) there follows

$$r^{\pi}(G) \equiv r^{\pi}(G/L)r^{\pi}(L/N)r^{\pi}(N)$$
$$\equiv r^{\pi}(G/N)r^{\pi}(N) \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}$$

since $d(|N|)/\text{g.c.d.}(|G|, d(|N|))$ divides $d(|L/N|)/\text{g.c.d.}(|G/N|, d(|L/N|))$, and $d(|G/N|)$ divides both $d(|G/L|)$ and $d(|L/N|)$.

Remark. The above proof is also valid for G/N a soluble group.

Theorem (2.12). *Let N be a normal subgroup of G. Then the following congruence holds:*

$$r^\pi(G)\equiv r^\pi(G/N)r^\pi(N) \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}.$$

PROOF. By using Lemma (2.11), the proof is analogous to that discussed for Theorem (2.3) (part (ii)) and is therefore omitted.

REMARKS

(1) If N is a normal subgroup of G and $r'^\pi(G)$ denotes the number of conjugacy classes of non-π-elements of G, then the following congruence holds:

$$(23)\qquad r'^\pi(G)\equiv r(G/N)r'^\pi(N)+r'^\pi(G/N) \pmod{d(|G|)\delta^\pi_{|G/N|}}.$$

Indeed, from Theorem (2.3) (part (ii)) we have

$$r^\pi(G)\equiv r^\pi(G/N)r^\pi(N) \pmod{d(|G|)\delta^\pi_{|G/N|}}$$

and putting $\pi=\pi(G)$ into the above congruence we get

$$r(G)\equiv r(G/N)r(N) \pmod{d(|G|)\delta_{|G/N|}};$$

but $\delta^\pi_{|G/N|}$ divides $\delta_{|G/N|}$, hence working modulo $d(|G|)\delta^\pi_{|G/N|}$ we have

$$(24)\qquad \begin{aligned} r'^\pi(G) &= r(G)-r^\pi(G)\\ &\equiv r(G/N)r(N)-r^\pi(G/N)r^\pi(N)\\ &= r(G/N)r^\pi(N)+r(G/N)r'^\pi(N)-r^\pi(G/N)r^\pi(N)\\ &= r'^\pi(G/N)\cdot r^\pi(N)+r(G/N)r'^\pi(N). \end{aligned}$$

In addition

$$r^\pi(G/N)\equiv |G/N|\equiv r(G/N) \pmod{\delta^\pi_{|G/N|}},$$

hence

$$r'^\pi(G/N)\equiv 0 \pmod{\delta^\pi_{|G/N|}}$$

and

$$r'^\pi(G/N)r^\pi(N)\equiv r'^\pi(G/N) \pmod{\delta^\pi_{|G/N|}d(|G|)}$$

inasmuch as $r^\pi(N)\equiv 1 \pmod{d(|G|)}$. Thus we conclude the required congruence.

(2) Let N be an abelian normal subgroup of G such that G/N is abelian. Then the following congruences hold:

(i) $r^\pi(G)\equiv |G|_\pi \pmod{d(|G|)\delta^\pi_{|G/N|}}$.

(ii) $r^\pi(G)\equiv |G|_\pi \pmod{d(|G/N|)\cdot d(|N|)/\text{g.c.d.}(|G|, d(|N|))}$.

Both congruences follow directly from Theorem (2.3) (part (ii)) and Theorem

(2.12), respectively (notice that $r^\pi(N) = |N|_\pi$, $r^\pi(G/N) = |G/N|_\pi$ and $|G|_\pi = |N|_\pi |G/N|_\pi$ in this case).

(3) As an immediate consequence of Theorem (2.3) (part (i))

$$r'^\pi(G) \equiv |G/N|\, r'^\pi(N) \pmod{\delta^\pi_{|G/N|}},$$

for each $N \trianglelefteq G$.

(4) In general the following congruence is not true:

$$r^\pi(G) \equiv |G| \pmod{d(|N|)/\text{g.c.d.}(|G|, d(|N|))}.$$

For example, if $G = (C_5 \times C_5) \times_f C_3$, $\pi = \{5\}$ and $N = C_5 \times C_5$, then we have $r^\pi(G) = r^\pi_G(N) = 9$, $d(|N|) = 4$ and $r^\pi(G) = 9 \not\equiv 75 \pmod 4$.

(5) In general the following congruence is not true:

$$r^\pi(G) \equiv r^\pi(G/N) r^\pi(N) \pmod{d(|N|)}.$$

In fact, if $G = \Sigma_3$, $N = C_3$ and $\pi = \{3\}$, we have $r^\pi(G) = r^\pi_G(N) = 2$, $d(|N|) = 2$ and $r^\pi(G) = 2 \not\equiv r^\pi(G/N) r^\pi(N) = 1 \cdot 3 \pmod 2$.

3. Supersoluble groups

In this section, the number of conjugacy classes in a supersoluble group is investigated by using the local properties of the number $r_G(gN)$.

LEMMA (3.1). *Let A, K and N be normal subgroups of M satisfying the following conditions: $A \leqq K \leqq N < M$, $A \leqq Z(M)$, $M/N = \langle \bar{x} \rangle \simeq C_q$, with q prime, M/K abelian and* g.c.d.$(|A|, |M/K|) = 1$. *Set $s = r_M(xN)$ and $s^* = r_{M/A}(\tilde{x}N/A)$. Then the following inequality holds:*

$$s - s^* \geqq (|A| - 1) \cdot |N/K|. \tag{25}$$

PROOF. Let us consider the following notation: $\bar{M} = M/N$, $\tilde{M} = M/A$ and $\tilde{\tilde{M}} = M/K$ (isomorphic to $(M/A)/(K/A)$). Let $\{\tilde{\tilde{x}}\tilde{\tilde{n}}_1, \ldots, \tilde{\tilde{x}}\tilde{\tilde{n}}_t\}$ be a complete system of representatives from distinct conjugacy $\tilde{\tilde{M}}$-classes that make up the coset $\tilde{\tilde{x}}\tilde{\tilde{N}}$ chosen so that $o(xn_i)$ and $|A|$ are relatively prime numbers for all $i = 1, \ldots, t$. By observing that $\tilde{x}\tilde{n}_i$ and $\tilde{x}\tilde{n}_j$ are not $\tilde{M}$-conjugate elements for all $i \neq j$ (because $\tilde{\tilde{M}}$ is an epimorphic image of $\tilde{M}$) we may consider a complete system of representatives $\{\tilde{x}\tilde{n}_1, \ldots, \tilde{x}\tilde{n}_t, \tilde{x}\tilde{n}_{t+1}, \ldots, \tilde{x}\tilde{n}_{s^*}\}$ from distinct conjugacy $\tilde{M}$-classes that intersect $\tilde{x}\tilde{N}$ ordered in the indicated form. From Lemma (1.1) we have

$$r_M(xN) = \sum_{i=1}^{s^*} r_M(xn_iA).$$

Further, for each $i \leqq t$ we have

$$r_M(xn_iA) = r_{C_M(xn_i)}(C_A(xn_i)) = |A|,$$

by using both Lemma (1.2) and A central in M. Thus $r_M(xN) \geqq s^* + (|A| - 1)t$. But $t = |N/K|$, since M/K is abelian, therefore we conclude the desired inequality.

LEMMA (3.2). *Let A, L be two normal subgroups of G such that $A \leqq Z(L)$ and G/L is abelian. Set $K/L = O_{\pi(A)}(C_G(A)/L)$. Then the following inequality holds*:

$$r(G) - r(G/A) \geqq (|A| - 1)\cdot(|G/C_G(A)| + 1/|G/K|)\cdot(|C_G(A)/K| - 1) + (r(K) - r(K/A))/|G/K|. \tag{26}$$

PROOF. Evidently we have $A \leqq L \leqq K \leqq C_G(A)$ and we may consider a series

(27) $K = N_0 < N_1 < \cdots < N_{t-1} < N_t = C_G(A) < N_{t+1} < \cdots < N_u = G$

(with $t = 0$, in case $K = C_G(A)$) chosen so that N_i is normal in G and $N_i/N_{i-1} = \langle \bar{x}_i \rangle \simeq C_{p_i}$, with p_i prime for all $i = 1, \ldots, u$ (that is possible, because G/K is an abelian group). Let

$$s_i = r_{N_i}(x_iN_{i-1}) \quad \text{and} \quad s_i^* = r_{N_i/K}(\bar{x}_iN_{i-1}/K) \qquad \text{for all } i = 1, \ldots, u.$$

Arguing as in Theorem of [7] the following equalities hold:

$$|G/K|\, r(G) = \sum_{i=1}^{u} (p_i^2 - 1)|N_{i-1}/K|\, s_i + r(K), \tag{28}$$

$$|G/K|\, r(G/A) = \sum_{i=1}^{u} (p_i^2 - 1)|N_{i-1}/K|\, s_i^* + r(K/A), \tag{29}$$

$$|G/K|^2 = \sum_{i=1}^{u} (p_i^2 - 1)|N_{i-1}/K|^2 + 1 \tag{30}$$

and

$$|G/C_G(A)|\,|G/K| = \sum_{j=t+1}^{u} (p_j^2 - 1)|N_{j-1}/K|^2 + |C_G(A)/K| \tag{31}$$

(notice that (30) and (31) are deduced by considering the following induced series:

$$\{\bar{1}\} = N_0/K < N_1/K < \cdots < N_u/K = G/K).$$

From (28) and (29) we get

$$\begin{aligned} |G/K|(r(G) - r(G/A)) = \sum_{i=1}^{u} (p_i^2 - 1)|N_{i-1}/K|(s_i - s_i^*) \\ + r(K) - r(K/A). \end{aligned} \tag{32}$$

Moreover, the following conditions hold:

$$\text{g.c.d.}(|A|, |N_i/K|) = 1 \quad \text{and} \quad A \leqq Z(N_i) \qquad \text{for all } i = 1, \ldots, t$$

(because G/L is abelian and $K/L = O_{\pi(A)}(C_G(A)/L)$), therefore by using Lemma (3.1) we get

$$s_i - s_i^* \geqq (|A| - 1)\cdot |N_{i-1}/K| \qquad \text{for all } i = 1, \ldots, t,$$

and consequently (32) yields

$$\begin{aligned} |G/K| \cdot (r(G) - r(G/A)) \geqq \sum_{i=1}^{t} (p_i^2 - 1)|N_{i-1}/K|^2(|A| - 1) \\ + r(K) - r(K/A). \end{aligned} \tag{33}$$

Finally, (30), (31) and (33) imply the required inequality.

LEMMA (3.3). *Let A be a cyclic normal subgroup of G of order a prime number. Then we have*

$$|G/C_G(A)|(r(G) - r(G/A)) = r(C_G(A)) - r(C_G(A)/A).$$

PROOF. Let us consider a series $C_G(A) = N_t < N_{t+1} < \cdots < N_u = G$ of normal subgroups of G having cyclic factors of order a prime number. By using the above notation (with $K = L = C_G(A)$) we have

$$\begin{aligned} |G/C_G(A)|(r(G) - r(G/A)) = \sum_{i=t+1}^{u} (p_i^2 - 1)|N_{i-1}/C_G(A)|(s_i - s_i^*) \\ + r(C_G(A)) - r(C_G(A)/A). \end{aligned}$$

Furthermore, if $\{\tilde{x}_i\tilde{n}_1, \ldots, \tilde{x}_i\tilde{n}_{s_i^*}\}$ is a complete system of representatives from distinct conjugacy N_i/A-classes that intersect $\tilde{x}_iN_{i-1}/A$, then we have

$$s_i = r_{N_i}(x_i N_{i-1}) = \sum_{j=1}^{s_i^*} r_{N_i}(x_i n_j A).$$

In addition, $x_i n_j$ is contained in $N_i - N_{i-1}$, hence $x_i n_j \notin C_G(A)$ and consequently from Lemma (1.10) it follows that $r_{N_i}(x_i n_j A) = 1$. Therefore $s_i = s_i^*$ and we conclude the required equality.

LEMMA (3.4). *Let A, L be two normal subgroups of G satisfying the following conditions: $A \leqq Z(L)$, $A \simeq C_p$, p prime, G/L abelian and $O_p(C_G(A)/L) \neq C_G(A)/L$. Then the following inequality holds:*

$$r(G) - r(G/A) \geqq 2^{\pi(|G|)} \cdot p.$$

PROOF. Let us use the inequality (26) for $\pi(A) = \{p\}$. Assume that $2 \nmid |G|$. If $C_G(A)$ is properly contained in G we have $r(G) - r(G/A) \geqq 6(p-1) \geqq 2p$. If $C_G(A) = G$ and $|G/K| > 3$ it is evident. If $|G/K| = 3$ we have $r(K) \geqq r(K/A) + p - 1$ (since $A \leqq Z(K)$) and consequently

$$r(G) - r(G/A) \geqq (p-1)(4/3)2 + (p-1)/3 \geqq 2p,$$

and the inequality has been proved when G has odd order. On the other hand, if G is a group of even order, the inequality also holds because

$$r(G) - r(G/A) \geqq p - 1 + (r(K) - r(K/A)/|G/K|) > p - 1$$

implies

$$r(G) - r(G/A) \geqq p.$$

Next we analyse the supersoluble groups of the type $G = P \times_\lambda C_t$, where P is a p-group of order p^{2n+e}, with $n \geqq 0$ and $e = 0$ or 1, and t a natural number dividing $p - 1$.

Let A be a normal subgroup of G such that $A \simeq C_p$. Then we have $C_G(A) = PC_{t'}$ for some t' dividing t, and by using Lemma (3.3)

$$(t/t') \cdot (r(G) - r(G/A)) = r(P \times_\lambda C_{t'}) - r(P/A \times_\lambda C_{t'})$$

is satisfied. Let us assume $t' = 1$ and $e = 0$ (that is, $C_G(A) = P$). By using P. Hall's formula (cf. [2]) it follows that there exist non-negative numbers k, k' such that

$$r(P) - r(P/A) = p(p-1) + (k - k')(p^2 - 1)(p - 1),$$

but $r(P) > r(P/A)$, hence necessarily $k - k'$ is non-negative, so $r(P) - r(P/A)$ is greater than or equal to $p(p-1)$ and we conclude

$$r(G) - r(G/A) \geqq p(p-1)/t \geqq 2^{\eta(|G|)} \cdot p.$$

Suppose that $t' \neq 1$. Putting $L = P$ into Lemma (3.4) we obtain

$$r(G) - r(G/A) \geqq 2^{\eta(|G|)} \cdot p.$$

However, in general, the above inequality is not true, in case $t' = 1$ and $e = 1$. For example, for $G = \mathrm{Hol}(C_9) = C_9 \times_\lambda C_6$ and $A = Z(P)$ the following data are true: $r(G) = 10$, $r(G/A) = r(C_3 \times \Sigma_3) = 9$ and $r(G) - r(G/A) = 1 \ngeq 3$. Neverthless, in this case, we have $|P/A| = p^{2n}$, hence there exists $C_p \simeq B/A \trianglelefteq G/A$ such that

$$r(G/A) \geqq r((G/A)/(B/A)) + 2^{\eta(|G|)} \cdot p$$

and consequently we may consider $B \trianglelefteq G$ satisfying $|B| = p^2$ and $r(G) \geqq r(G/B) + 2^{\eta(|G|)} p$ (if $n \geqq 1$).

In the following we show the above inequalities for any supersoluble group.

LEMMA (3.5). *Let A, L, be two normal subgroups of G such that $A \leqq L$ and G/L is abelian. Then we have*

$$|G/L|(r(G) - r(G/A)) \geqq r(L) - r(L/A).$$

In particular, if A is cyclic of prime order and $L = C_G(A)$, then the bound is attained.

PROOF. By considering a series $L = N_0 < N_1 < \cdots < N_v = G$ such that $N_i \trianglelefteq G$ and $|N_i/N_{i-1}|$ is prime for all i, and arguing as in Lemma (3.3), this result follows immediately.

In the following, $F(G)$ denotes the Fitting subgroup of G, that is, the unique maximal normal nilpotent subgroup of G.

THEOREM (3.6). *Let G be a supersoluble group such that $F(G)$ is not cyclic of prime order. Then one of the following conditions is satisfied:*

(1) *There exists p prime and $A \simeq C_p$ normal in G such that*

$$r(G) \geqq r(G/A) + 2^{\eta(|G|)} p.$$

(2) *There exists p prime such that $|F(G)| = p^{2n+1}$ for some $n \geqq 1$, and there exists $B \trianglelefteq G$ such that $|B| = p^2$ and $r(G) \geqq r(G/B) + 2^{\eta(|G|)} p$.*

PROOF. Let us assume that $F(G)$ is p-group for some prime p. Since G is supersoluble, $G/F(G)$ is an abelian group and consequently p does not divide $|G/F(G)|$. Let A be a normal subgroup of G isomorphic to C_p. Evidently

$O_p(C_G(A)/F(G))$ is trivial. If $C_G(A) = F(G)$ then we have $G = F(G) \times_\lambda C_t$ for some t dividing $p-1$ and the result follows from the previous commentary to the above lemma. On the other hand, if $C_G(A) > F(G)$, then by using Lemma (3.4) with $L = F(G)$, it follows that

$$r(G) - r(G/A) \geqq 2^{\eta(|G|)} p$$

and the theorem is proved. Thus, we can suppose that $F(G)$ is a nilpotent group of order divisible by at least two prime numbers p and q, with $q < p$. Let A_1, A_2 be two normal subgroups of G such that $A_1 \simeq C_p$ and $A_2 \simeq C_q$. If

$$O_p(C_G(A_1)/F(G)) \neq C_G(A_1)/F(G) \quad \text{or} \quad O_q(C_G(A_2)/F(G)) \neq C_G(A_2)/F(G),$$

then again (1) follows by using Lemma (3.4). Therefore we also can assume that $C_G(A_1)/F(G)$ is a p-group and $C_G(A_2)/F(G)$ is a q-group. Since $|G/C_G(A_2)|$ divides $q-1$ and q is less than p, it follows that p does not divide $|G/F(G)|$ and consequently

$$O_p(C_G(A_1)/F(G)) = 1 = C_G(A_1)/F(G).$$

Now, bearing in mind Lemma (3.5) with $L = F(G)$ we have

$$|G/F(G)| \cdot (r(G) - r(G/A_2)) \geqq r(F(G)) - r(F(G)/A_2).$$

Let P (resp. Q) be the Sylow p-subgroup (resp. Sylow q-subgroup) of $F(G)$ and let T be the normal subgroup of G such that $F(G) = P \times Q \times T$. Then we have

$$r(F(G)) - r(F(G)/A_2) = r(P)r(T)(r(Q) - r(Q/C_q)) \geqq p(q-1)$$

and consequently

$$r(G) - r(G/A_2) \geqq p(q-1)/|G/F(G)| \geqq 2^{\eta(|G|)} \cdot (p(q-1)/(p-1)) > 2^{\eta(|G|)}(q-1)$$

(because $|G/F(G)|$ divides $p-1$), so $r(G) - r(G/A_2) \geqq 2^{\eta(|G|)}(q-1) + 1$. By using Theorem (2.12) in case $|G|$ odd, we get $r(G) - r(G/A_2) \equiv 0$ $(\text{mod}(q-1)/\text{g.c.d.}(|G|, q-1))$, hence necessarily $r(G) - r(G/A_2) \geqq 2q$ and the Theorem is proved.

Next we analyse the case $|F(G)| = p$, with p a prime number. In this case we have:

LEMMA (3.7). *Let G be a supersoluble group of order*

$$pm = p \cdot \prod_{i=1}^{t} p_i^{2n_i + e_i},$$

with p a prime number not dividing m, $p_i \neq p_j$ for each $i \neq j$, $e_i = 0$ or 1 and p_i prime for all $i = 1, \ldots, t$. Suppose $|F(G)| = p$. Then the following inequalities hold:

(1) *If* $|G|$ *is odd*,

$$r(G) \geqq \sum_{n_i \neq 0} (2n_i + e_i) p_i + \sum_{n_i = 0} (p_i - 1)^{1/2} + (p - 1)^{1/2}.$$

(2) *If* $|G|$ *is even*,

$$r(G) \geqq \sum_{n_i \neq 0} (2n_i + e_i - 1) p_i + \sum_{n_i = 0} (p_i - 1)^{1/2} + (p - 1)^{1/2}.$$

Proof. Evidently, G is a Frobenius group of kernel $F(G)$ and complement isomorphic to one cyclic group of order m. Thus we have $r(G) = m + (p - 1)/m$. We consider the following function:

$$f(m) = \begin{cases} \sum_{n_i \neq 0} (2n_i + e_i) p_i + \sum_{n_i = 0} (p_i - 1)^{1/2} & \text{if } m \text{ is odd,} \\ \sum_{n_i \neq 0} (2n_i + e_i - 1) p_i + \sum_{n_i = 0} (p_i - 1)^{1/2} & \text{if } m \text{ is even.} \end{cases}$$

It can easily be shown that the relation $m \geqq 2 \cdot f(m)$ holds for each natural $m \geqq 2$. We see that $r(G) = m + (p - 1)/m$ is greater than or equal to $f(m) + (p - 1)^{1/2} = f(mp)$. Suppose that $m \leqq (p - 1)^{1/2}$. In this case we have

$$r(G) = m + (p - 1)/m \geqq f(m) + (p - 1)/(p - 1)^{1/2} = f(m) + (p - 1)^{1/2}.$$

Therefore we may assume that m is greater than $(p - 1)^{1/2}$. Suppose the result false, that is,

$$m + (p - 1)/m < f(m) + (p - 1)^{1/2},$$

then we have

$$m < f(m) - (p - 1)/m + (p - 1)^{1/2}$$

and necessarily $f(m) - (p - 1)/m$ is greater than 0. Further, $2f(m) \leqq m$ implies $f(m) < (p - 1)^{1/2}$, hence $m < 2(p - 1)^{1/2}$ and consequently

$$\begin{aligned} r(G) &= m + (p - 1)/m \geqq 2f(m) + (p - 1)/m \\ &> f(m) + 2(p - 1)/m > f(m) + 2(p - 1)/2(p - 1)^{1/2} \\ &= f(m) + (p - 1)^{1/2} \end{aligned}$$

a final contradiction, therefore the inequality is true.

Theorem (3.8). *Let* G *be a supersoluble group of order* $|G| = \prod_{i=1}^{t} p_i^{2n_i + e_i}$, *with* $p_i \neq p_j$ *for each* $i \neq j$, $n_i \geqq 0$, $e_i = 0$ *or* 1, *and* p_i *prime for all* $i = 1, \ldots, t$. *Then the following inequality holds*:

$$r(G) \geqq \sum_{n_i \neq 0} 2^{\eta(|G|)} n_i p_i + \sum_{n_i = 0} (p_i - 1)^{1/2}.$$

PROOF. We argue by induction on $|G|$. If $|F(G)| = p$, with p prime, the result follows from Lemma (3.7). Thus we can suppose that $F(G)$ is not cyclic of order prime. Now bearing in mind Theorem (3.6) we have two possibilites:

(1) There exists $A \simeq C_p$ normal in G such that $r(G) \geqq r(G/A) + 2^{\eta(|G|)} p$. Assume that p^{2n+e} is the greatest power of p dividing $|G|$. We now apply induction to supersoluble quotient group G/A and evidently we only need to analyse the summands relative to prime p in the final summation. If $p \parallel |G|$, it is $2^{\eta(|G|)} p > (p-1)^{1/2}$; if $p^2 \parallel |G|$ it is $(p-1)^{1/2} + 2^{\eta(|G|)} p > 2^{\eta(|G|)} p$; finally, if $2n + e \geqq 3$ we have

$$2^{\eta(|G|)}(n + e - 1) p + 2^{\eta(|G|)} p \geqq 2^{\eta(|G|)}(n + e) p.$$

(2) $|F(G)| = p^{2n+1}$ with $n \geqq 1$, and there exists $B \trianglelefteq G$ of order p^2 and satisfying $r(G) \geqq r(G/B) + 2^{\eta(|G|)} p$. In this case, we apply induction to supersoluble group G/B. Clearly $p^{2n+1} \parallel |G|$. If $n = 1$, then $p \parallel |G/B|$ and $(p-1)^{1/2} + 2^{\eta(|G|)} p > 2^{\eta(|G|)} p$. If $n \geqq 2$, we have

$$2^{\eta(|G|)}(n - 1) p + 2^{\eta(|G|)} p \geqq 2^{\eta(|G|)} n p.$$

Thus, our inductive argument yields the desirable inequality, in any case.

Evidently the bound obtained in the above Theorem improves M. Cartwright's bound given in [1].

Next we analyse the residue class of $r(G)$, modulo the "best" number given in terms of the primes dividing $|G|$.

LEMMA (3.9). *Let $a_1, \ldots, a_t$, m be natural numbers such that $a_i \equiv 1$ (mod m) for all $i = 1, \ldots$. Then the following congruence holds*:

$$\prod_{i=1}^{t} a_i \equiv \sum_{i=1}^{t} a_i - (t - 1) \pmod{m^2}.$$

PROOF. We argue by induction on t. In case $t = 2$ we have

$$a_1 a_2 = (a_1 - 1)(a_2 - 1) + a_1 + a_2 - 1 \equiv a_1 + a_2 - 1 \pmod{m^2}.$$

Suppose $t \geqq 3$. We have $a_2 \cdots a_t \equiv 1 \pmod{m}$, hence our induction argument yields

$$\prod_{i=1}^{t} a_i \equiv a_1 + (a_2 \cdots a_t) - 1 \equiv a_1 + \sum_{i=2}^{t} a_i - (t - 2) - 1 \pmod{m^2}.$$

Let G be a finite group of order $\Pi_{i=1}^{t} p_i^{2n_i+e_i}$. Evidently we have $p_i \equiv 1$ (mod $d(|G|)$), hence by using Lemma (3.9) we conclude that

$$|G| \equiv \sum_{i=1}^{t} (2n_i + e_i)(p_i - 1) + 1 \pmod{d(|G|)^2}. \tag{34}$$

On the other hand, we have $r(G) \equiv |G|$ (mod $d(|G|)^2$) (cf. [5]) hence $r(G)$ is congruent to the above summation modulo the number $d(|G|)^2$.

Now we may prove

THEOREM (3.10). *Let G be a supersoluble group of odd order $|G| = \Pi_{i=1}^{t} p_i^{2n_i+e_i}$, with $p_i \neq p_j$ for each $i \neq j$, p_i prime, n_i a non-negative integer number and $e_i = 0$ or 1. Then the number of conjugacy classes of G is of the form*

$$1 + \sum_{i=1}^{t} (2n_i(p_i - 1) + e_i\mu_{p_i}) + k \cdot d(|G|)^2$$

for some non-negative integer number k. In particular, G has at least

$$1 + \sum_{i=1}^{t} (2n_i(p_i - 1) + e_i\mu_{p_i})$$

conjugate classes.

PROOF. It follows easily from (34) that we only need prove the inequality

$$r(G) \geqq 1 + \sum_{i=1}^{t} (2n_i(p_i - 1) + e_i\mu_{p_i}).$$

We argue by induction on $|G|$, reasoning in a similar way to that in Theorem (3.8). If $|F(G)| = p$, p prime, then we have $G = C_p \times_f C_m$, $r(G) = m + (p-1)/m = m + \mu_p$ and the inequality is trivial.

Assume that $F(G)$ is not cyclic of prime order. By using Theorem (3.6) we may restrict ourselves to one of the following conditions:

(1) There exists $A \simeq C_p$ normal in G such that $r(G) \geqq r(G/A) + 2p$. Let p^{2n+e} be the power of p occurring in $|G|$. By applying inductive hypothesis to supersoluble group G/A we only need to analyse the summands relatives to prime p (because if q is a prime dividing $|G|$ and $q \neq p$, then it is $\mu_q(|G/A|) \geqq \mu_q(|G|)$). If $p \parallel |G|$, we have $2p > \mu_q(|G|)$; if $p^2 \parallel |G|$ we have $2p + \mu_p > 2(p-1)$; if $p^3 \parallel |G|$ we have $2p + 2(p-1) > 2(p-1) + \mu_p$; if $p^4 \parallel |G|$ we have $2p + 2(p-1) + \mu_p > 4(p-1)$; and finally, if $2n + e \geqq 5$ we have

$$2p + 2(n + e - 1)(p - 1) + (1 - e)\mu_p \geqq 2n(p - 1) + e\mu_p.$$

(2) $|F(G)| = p^{2n+1}$, with $n \geqq 1$, and there exists $B \trianglelefteq G$ such that $|B| = p^2$ and $r(G) \geqq r(G/B) + 2p$. Again we apply the inductive hypothesis to G/B. If $p^3 \parallel |G|$, then $p \parallel |G/B|$ and we have $\mu_p + 2p \geqq 2(p - 1) + \mu_p$; if $2n + e \geqq 5$ we have $2(n - 1)(p - 1) + \mu_p + 2p > 2n(p - 1) + \mu_p$. In any case, we obtain the desired inequality.

REMARK. Let p, q be two odd prime numbers such that q divides $p - 1$. Then the group $\mathrm{Hol}(C_p, C_q)$ attains the bound given in (3.10). Thus our results are in this case best possible.

THEOREM (3.11). *Let G be a supersoluble finite group and let p be the smaller prime number dividing $|G|$. Assume that $p^a \parallel |G|$. Then there exists a non-negative number k such that*

$$p^a \cdot r(G) = (p + 1)(p^a - 1) + r(O_{p'}(G)) + k \cdot (p^2 - 1) \cdot d(|G|).$$

PROOF. Let P be a Sylow p-subgroup of G. Since G is supersoluble we have $G = O_{p'}(G) \times_\lambda P$. Evidently, there exists a series

$$O_{p'}(G) = N_0 < N_1 < \cdots < N_a = G$$

satisfying $N_i \trianglelefteq G$ and $N_i/N_{i-1} = \langle \bar{x}_i \rangle \simeq C_p$ for all $i = 1, \ldots, a$. Set $s_i = r_{N_{i+1}}(x_i N_i)$. Arguing as in Theorem of [7] we have

$$p^a \cdot r(G) = \sum_{i=1}^{a} (p^2 - 1) p^{a-i} s_{a-i} + r(O_{p'}(G)).$$

Furthermore $s_{a-i} = 1 + k_{a-i} \cdot d(|G|)$ for some $k_{a-i} \geqq 0$. Therefore we conclude the desirable equality.

In particular, if G is a supersoluble group of even order, then we have

$$2^a \cdot r(G) = 3(2^a - 1) + r(O_{2'}(G)) + k \cdot 3,$$

where $O_{2'}(G)$ is a supersoluble group of odd order. Thus the number $r(O_{2'}(G))$ can be estimated by using Theorem (3.10).

4. The number $r_G(N)$

In this section we analyse the number of conjugacy classes that make up the normal subgroup N.

THEOREM (4.1). *Let N be a normal subgroup of G. Then there exists a non-negative integer number k such that*

$$r^{\pi}(G) = r^{\pi}_{G}(N) + r^{\pi}(G/N) - 1 + k \cdot d(|G|)d(|G/N|).$$

PROOF. Let us consider the following decomposition:

$$[G/N]_{\pi} = \{\bar{1}\} \dot{\cup} \dot{\bigcup_{j=1}^{u}} \bigcup_{i=1}^{d(|G/N|)} \mathrm{Cl}_{\bar{G}}(\bar{g}_j^{\ell^i}),$$

where $\ell = \ell(G/N)$ and $\bar{G} = G/N$. Then we have $r^{\pi}(\bar{G}) = 1 + u \cdot d(|\bar{G}|)$ and

$$r^{\pi}(G) = r^{\pi}_{G}(N) + \sum_{j=1}^{u} d(|\bar{G}|) r^{\pi}_{G}(g_j N).$$

On the other hand, for each π-element $\bar{g}$ of $\bar{G}$ we have

$$r^{\pi}_{G}(gN) = 1 + k_g d(|G|) \qquad \text{for some } k_g \geqq 0,$$

hence we conclude the desirable equality.

In case $\pi = \pi(G)$ we can improve the above equality. Indeed, for each element $g \in G$ we have

$$r_G(gN) = |N/(N \cap N_G(gN)')| + k'_g \cdot d(|N_G(gN)|)$$

for some non-negative integer k'_g (cf. [12] Corollary (3.21)). Furthermore

$$|N/(N \cap N_G(gN)')| = |N/(N \cap G')|\,|(N \cap G')/(N \cap N_G(gN)')|$$
$$\equiv |N/(N \cap G')| \pmod{d(|G|)},$$

hence there exists $k''_g \geqq 0$ so that

$$r_G(gN) = |N/(N \cap G')| + k''_g \cdot d(|G|).$$

Thus we have

$$r(G) = r_G(N) + \sum_{j=1}^{u} d(|\bar{G}|) \cdot r_G(g_j N)$$
$$= r_G(N) + |N/(N \cap G')|(r(G/N) - 1)$$
$$+ k \cdot d(|G|)d(|\bar{G}|) \qquad \text{for some } k \geqq 0$$

and we have shown the following result:

THEOREM (4.2). *Let N be a normal subgroup of G. Then there exists a non-negative number k such that*

(35) $r(G) = r_G(N) + |N/(N \cap G')|(r(G/N) - 1) + k \cdot d(|\tilde{G}|) \cdot d(|G|)$.

EXAMPLES

(1) Assume that G is a finite p-group for some prime number p. In this case, there exists $k \geqq 0$ such that

$$r(G) = r_G(N) + |N/(N \cap G')|(r(G/N) - 1) + k \cdot (p-1)^2.$$

In particular, putting $N = Z(G)$ in (35) we get

$$r(G) = |Z(G)| + |Z(G)/(Z(G) \cap G')|(r(G/Z(G)) - 1) + k \cdot (p-1)^2,$$

and putting $N = \phi(G)$ in (35) (where $\phi(G)$ is the Frattini subgroup of G), then we have

$$r(G) = r_G(\phi(G)) + |\phi(G)/G'|(|G/\phi(G)| - 1) + k \cdot (p-1)^2,$$

since $G' = [G, G]$ is contained in $\phi(G)$.

(2) Let N be a normal subgroup of a finite p-group G such that $N \leqq \phi(G)$ and G/N is abelian of type $(p^{t_1}, \ldots, p^{t_s})$. Then we have $\phi(G/N) = \phi(G)/N$ and clearly $\phi(G/N)$ is of type $(p^{t_1-1}, \ldots, p^{t_s-1})$, where

$$G/\phi(G) \simeq C_p \times \overset{s}{\cdots} \times C_p.$$

We have

$$|\phi(G)/G'| = |\phi(G)/N|\,|N/G'| = p^{(t_1+\cdots+t_s)-s} \cdot |N/G'|$$

and consequently there exists $k \geqq 0$ such that

$$r(G) = r_G(\phi(G)) + p^{(t_1+\cdots+t_s)-s}|N/G'|(p^s - 1) + k \cdot (p-1)^2.$$

(3) Let us consider the upper central series of a finite p-group G:

$$1 = Z_0 < Z_1 < \cdots < Z_{c-1} < Z_c = G,$$

where $Z_1 = Z(G)$ and $Z_{i+1} = Z_{i+1}(G)$ is defined by $Z_{i+1}/Z_i = Z(G/Z_i)$ for each $i = 1, \ldots, c-1$. The number $c - 1$ is called the nilpotent class of G. Evidently we have

$$r(G) = |Z(G)| + \sum_{i=2}^{c} r_G(Z_i - Z_{i-1})$$

and

$$r_G(Z_i - Z_{i+1}) = \sum_{x \in Z_i/Z_{i-1} - \{1\}} r_G(xZ_{i-1})$$

$$= (|Z_i/Z_{i-1}| - 1)(|Z_{i-1}/(Z_{i-1} \cap G')|)$$

$$+ k_i \cdot (p-1)^2 \qquad \text{for some } k_i \geqq 0.$$

Thus we conclude

$$r(G) = |Z(G)| + \sum_{i=2}^{c} (|Z_i/Z_{i-1}| - 1)|Z_{i-1}/(Z_{i-1} \cap G')| + k' \cdot (p-1)^2 \tag{36}$$

for some $k' \geqq 0$.

Let N be a normal subgroup of a finite p-group G and assume that $|N| = p^{m'} = p^{2n'+e'}$ with $n' \geqq 0$ and $e' = 0$ or 1. We have $r_G(N) \equiv |N|$ $(\mathrm{mod}\,(p-1)^2)$ and

$$|N| \equiv n'(p^2 - 1) + p^{e'} \equiv 2n'(p-1) + p^{e'} \pmod{(p-1)^2}.$$

Thus we have

$$r_G(N) \equiv 2n'(p-1) + p^{e'} \pmod{(p-1)^2}.$$

Let us consider a series $1 = N_0 < N_1 < \cdots < N_{m'} = N$ such that $N_i \trianglelefteq G$ and $N_i/N_{i-1} = \langle \bar{x}_i \rangle \simeq C_p$ for all i. Then we have

$$r_G(N) = |N_1| + \sum_{i=2}^{m'} r_G(x_i N_{i-1})(p-1)$$

and

$$r_G(x_i N_{i-1}) = |N_{i-1}/(N_{i-1} \cap G')| + k_i(p-1)^2 \qquad \text{for some } k_i \geqq 0.$$

Therefore

$$r_G(N) = p + (p-1) \sum_{i=2}^{m} |N_{i-1}/(N_{i-1} \cap G')| + k' \cdot (p-1)^2 \qquad \text{with } k' \geqq 0.$$

In particular $r_G(N) \geqq (m-1)(p-1) + p^{e'}$. Furthermore, equality holds if and only if $r_G(x_i N_{i-1}) = 1$ for all i, that is, $x_i N_{i-1} = \mathrm{Cl}_G(x_i)$ for all i. In this case we have $|C_G(x_i)| = |G/N_{i-1}|$ and

$$\Delta_N^G = (|G|, \overset{p-1}{\ldots\ldots}, |G|, \ldots, |G/N_{m-1}|, \overset{p-1}{\ldots\ldots}, |G/N_{m-1}|).$$

This latter information gives us

THEOREM (4.3). *Let N be a normal subgroup of a finite p-group G and suppose that $|N| = p^{2n'+e'}$ with $n' \geqq 0$ and $e' = 0$ or 1. Then there exists a non-negative integer number k such that*

$$r_G(N) = 2n'(p-1) + p^{e'} + k\cdot(p-1)^2.$$

Further $r_G(N) = 2n'(p-1) + p^{e'}$ if and only if

$$\Delta_N^G = (|G|, \overset{p-1}{\ldots\ldots}, |G|, \ldots, |G/N|\,p, \overset{p-1}{\ldots\ldots}, |G/N|\,p).$$

REMARK. Let G be a non-abelian group of order p^3 and let N be a normal subgroup of order p^2. Then we have $r_G(N) = 2(p-1) + 1$ and

$$\Delta_N^G = (p^3, \overset{p-1}{\ldots\ldots}, p^3, p^2, \overset{p-1}{\ldots\ldots}, p^2).$$

Let G be a finite p-group and let N, M be two normal subgroups of G such that $M \leqq N$ and $N/M \leqq Z(G/M)$. Then

$$r_G(N-M) = (|N/M| - 1)|M/(M \cap G')| + k\cdot(p-1)^2 \qquad \text{for some } k \geqq 0.$$

In particular $r_G(N-M) \geqq |N/M| - 1$ and $r_G(N-M) = |N/M| - 1$ if and only if

$$\Delta_{N-M}^G = (|G/M|, \overset{|N/M|-1}{\ldots\ldots\ldots}, |G/M|).$$

We have

THEOREM (4.4). *Let N be a normal subgroup of a finite p-group G and let α be the greatest subscript such that $N \not\subseteq Z_\alpha$ (hence $N \subseteq Z_{\alpha+1}$). Then the following assertions hold:*

(i) $|N| \geqq p^{\alpha+1}$.

(ii) $r_G(N) = |N \cap Z_1| + \Sigma_{j=1}^{\alpha} |(N \cap Z_{j+1})/(N \cap Z_j)| - \alpha + k\cdot(p-1)^2$ *for some $k \geqq 0$. Furthermore, $k = 0$ if and only if*

$$\Delta_N^G = (|G|, \overset{|N\cap Z_1|}{\ldots\ldots}, |G|;\ |G/(N\cap Z_1), \overset{|(N\cap Z_2)/(N\cap Z_1)|-1}{\ldots\ldots\ldots\ldots}, |G/(N\cap Z_1)|;$$

$$\ldots;\ |G/(N\cap Z_\alpha)|, \overset{|(N\cap Z_{\alpha+1})/(N\cap Z_\alpha)|-1}{\ldots\ldots\ldots\ldots}, |G/(N\cap Z_\alpha)|).$$

If $k \geqq 1$, then we have

$$r_G(N) \geqq |N\cap Z_1| + \sum_{j=1}^{\alpha} |(N\cap Z_{j+1})/(N\cap Z_j)| - \alpha + (p-1)^2$$

$$\geqq (\alpha+1)|N|^{1/(\alpha+1)} - \alpha + (p-1)^2.$$

PROOF. Since N is not contained in Z_j for each $j \leqq \alpha$ we have $N \cap Z_j < N$ and there exists $N_j \trianglelefteq G$ such that $N_j \leqq N$ and $N_j/(N \cap Z_j) \simeq C_p$. Therefore $N_j/(N \cap Z_j)$ is a subgroup of $(N \cap Z_{j+1})/(N \cap Z_j)$ and $(N \cap Z_{j+1}) - (N \cap Z_j)$ is a non-empty normal subset of G for every $j \leqq \alpha$. Now, let us consider the chain

$$1 < N \cap Z_1 < N \cap Z_2 < \cdots < N \cap Z_{\alpha+1} = N.$$

Then necessarily $|N| \geqq p^{\alpha+1}$ and

$$r_G(N) = |N \cap Z_1| + \sum_{j=1}^{\alpha} r_G(N \cap Z_{j+1} - (N \cap Z_j)),$$

where $(N \cap Z_{j+1})/(N \cap Z_j) \leqq Z(G/(N \cap Z_j))$. Thus, all the assertions are a direct consequence of the previous commentary to (4.4) and from the fact that the arithmetic mean of $\alpha + 1$ numbers is larger than or equal to the geometric mean of such numbers, that is,

$$(\alpha+1)(1/(\alpha+1))\left(|N \cap Z_1| + \sum_{j=1}^{\alpha} |(N \cap Z_{j+1})/(N \cap Z_j)|\right)$$

$$\geqq (\alpha+1)\left(\prod_{j=1}^{\alpha} |N \cap Z_1| \cdot (|N \cap Z_{j+1}|)/|N \cap Z_j|)\right)^{1/(\alpha+1)}$$

$$= (\alpha+1)|N|^{1/(\alpha+1)}.$$

COROLLARY (4.5). *For each finite p-group G, either*

$$\Delta_G = (|G|, \overset{|Z_1|}{\ldots\ldots}, |G|; |G/Z_1|, \overset{|Z_2/Z_1|-1}{\ldots\ldots\ldots}, |G/Z_1|;$$

$$\ldots; |G/Z_{c-1}|; \overset{|G/Z_{c-1}|-1}{\ldots\ldots\ldots}, |G/Z_{c-1}|)$$

or

$$r(G) \geqq c|G|^{1-c} - (c-1) + (p-1)^2,$$

where $c - 1$ is the nilpotent class of G.

(This result refines Sherman's inequality given in [6].)

THEOREM (4.6). *Let i be the first subscript such that $Z_i \not\subseteq G'$ and suppose that G is a finite p-group of nilpotent class $c - 1$. Then the following inequality holds*:

$$r(G) \geqq c(|G|p^{c-i-1})^{1/c} - (c-i-1)p - i + 1.$$

PROOF. By using (36) we have

$$r(G) \geqq \sum_{j=i+1}^{c} (|Z_j/Z_{j-1}| - 1)p + \sum_{j=1}^{i} (|Z_j/Z_{j-1}| - 1) + 1$$

$$= \sum_{j=i+1}^{c} |Z_j/Z_{j-1}|\, p + \sum_{j=1}^{i} |Z_j/Z_{j-1}| - (c-i-1)p - i + 1$$

$$\geqq c\left(\prod_{j=i+1}^{c} (|Z_j/Z_{j-1}|\, p) \prod_{k=1}^{i} |Z_k/Z_{k-1}| \right)^{1/c} - (c-i-1)p - i + 1$$

$$= c(|G|\, p^{c-i-1})^{1/c} - (c-i-1)p - i + 1.$$

REMARK. If $Z(G) \not\subseteq G'$, then the following inequality holds:

$$r(G) \geqq c(|G|\, p^{c-2})^{1/c} - (c-2)p.$$

Now let us consider the lower central series $G > Y_2 > \cdots > Y_c = 1$ of a finite p-group G, where

$$Y_2 = G' \quad \text{and} \quad Y_{i+1} = [Y_i, G] = \langle [x,y] \mid x \in Y_i, y \in G \rangle.$$

Let N be a normal subgroup of G and let β be the greatest subscript such that $Y_\beta \not\subseteq N$. Then we have the following chain:

$$G = NY_0 > NY_1 > NY_2 > \cdots > NY_\beta > NY_{\beta+1} = N$$

and consequently $|G/N| \geqq p^{\beta+1}$. Further

$$r_G(G - N) = \sum_{i=1}^{\beta+1} r_G(NY_{i-1} - NY_i)$$

where $NY_{i-1}/NY_i \leqq Z(G/NY_i)$. Thus we can establish a similar result to Theorem (4.4):

THEOREM (4.7). *Let G be a finite p-group and let N be a normal subgroup of G. Let β be the greatest subscript such that $Y_\beta \not\subseteq N$. Then we have the following affirmations*:

(i) $|G/N| \geqq p^{\beta+1}$.

(ii) $r_G(G-N) = \sum_{i=1}^{\beta+1}(|NY_{i-1}/NY_i| - 1) + k\cdot(p-1)^2$ *for some* $k \geqq 0$.

(iii) $r_G(G-N) \geqq (\beta+1)\cdot |G/N|^{1/(\beta+1)} - (\beta+1)$.

(iv) *Either* $r_G(G-N) = \sum_{i=1}^{\beta+1}(|NY_{i-1}/NY_i| - 1)$, *where*

$$\Delta_{G-N}^{G} = (|G/N|, \overset{|NY_\beta/N|-1}{\ldots\ldots\ldots}, |G/N|; |G/NY_\beta|, \overset{|NY_{\beta-1}/NY_\beta|-1}{\ldots\ldots\ldots}, |G/NY_\beta|;$$

$$\ldots; |G/NY_1|, \overset{|G/NY_1|-1}{\ldots\ldots\ldots}, |G/NY_1|),$$

or

$$r_G(G-N) \geqq \sum_{i=1}^{\beta+1} (|NY_{i-1}/NY_i| - 1) + (p-1)^2$$

$$\geqq (\beta+1)\cdot|G/N|^{1/(\beta+1)} - (\beta+1) + (p-1)^2.$$

REMARK. Putting $N = 1$ in (iii), we again get Sherman's bound given in [6], and in (iv) we improve it.

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